PHYSICAL REVIEW LETTERS

VOLUME 84 | 28 FEBRUARY 2000 | NUMBER 9

Nonclassical States: An Observable Criterion

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(Received 10 March 1999; revised manuscript received 28 July 1999)

An observable criterion is derived that allows one to distinguish nonclassical states of the harmonic oscillator from those having a classical counterpart. A quantum state is shown to have no classical counterpart if and only if the characteristic functions of the quadrature distributions or the s-parametrized phase-space distributions exhibit a slower decay than for the ground state of the oscillator. This renders it possible to experimentally check the failure of the P function to be a probability measure.

PACS numbers: 03.65.Bz, 42.50.Ar, 42.50.Dv

The study of nonclassical states of light has been a subject of rapidly increasing interest. Since the pioneering demonstration of photon antibunching by Kimble, Dagenais, and Mandel [1], there has been a manyfold of realizations of nonclassical effects of light, including sub-Poissonian photon statistics [2] and squeezing [3]. Nonclassical states could also be prepared in cavity QED [4] and in matter systems such as atomic Rydberg wave packets [5], molecular vibrations [6], and trapped atoms [7].

The formulation of a reasonable criterion for a quantum state to be considered as a nonclassical one has attracted a great deal of interest ranging from the 1960s until now [8–10]. The most common criterion is based on the P distribution of Glauber and Sudarshan [11]. Following Titulaer and Glauber, "fields with positive definite P functions... are, in fact, precisely the quantum fields which may be described in a natural way as possessing classical analogs" [8]. In the same sense Mandel has stated, "if P is not a probability density, then the state is nonclassical" [9].

From the theoretical point of view this criterion includes the nonclassical effects that have been considered so far. However, in practice, one needs a criterion that allows one to distinguish between classical and nonclassical states in experiments. This can hardly be based on the P function that cannot be obtained from measurements. The P distribution may not only attain negative values, but it may even become highly singular. Criteria used in experiments are typically based on the variance or normally ordered variance of particular observables. The violation of such a criterion for a certain nonclassical property cannot be used to conclude that the quantum state under study has a classical counterpart. What one needs is a criterion that is equivalent to the one based on the P distribution and that is observable.

In the present contribution we introduce a sufficient and necessary criterion for a quantum state to be nonclassical. It allows one to experimentally check the failure of the P function to be a probability density. The criterion is related to the variables of classical theory, such as the trajectory for a mechanical oscillator and the field strength for a radiation field. We will confine our treatment to a single degree of freedom of a harmonic system. Although generalizations to two or more modes appear to be straightforward, we will not consider here specific nonclassical features such as entanglement.

For a unified treatment of mechanics and electrodynamics we use the phase-sensitive quadrature operator,

$$\hat{x}(\varphi) = \hat{a}e^{i\varphi} + \hat{a}^{\dagger}e^{-i\varphi}, \tag{1}$$

with $\hat{a}$ ($\hat{a}^{\dagger}$) being the annihilation (creation) operator of the harmonic mode. By identifying the phase parameter φ with νt, ν being the frequency of the oscillator, $\hat{x}(\varphi)$ describes the free evolution of the system. The operator $\hat{x}(\varphi)$ is scaled such that its spread in the ground state of the oscillator (or vacuum state in the case of a radiation mode) is unity, $(\Delta x)_{\mathrm{gr}} = 1$. That is, this operator may represent physical quantities such as position or momentum in mechanics and electric or magnetic field strengths in electrodynamics by simply multiplying $\hat{x}(\varphi)$ with the spread of the desired quantity in the ground state.

Besides the very close relation of the operator $\hat{x}(\varphi)$ to variables used in classical physics there are two other reasons to favor its use for the formulation of a criterion for nonclassical states. First, it is known that the statistics of this observable, for φ values in an interval of size π, represents the full information on the quantum state [12]. Second, the desired statistical distributions are measurable both for radiation fields [13] and for various quantum mechanical systems [14] (for a detailed review of the topic see [15]). Thus a criterion that is based on the observable $\hat{x}(\varphi)$ would exhibit two advantages: it is measurable and it completely characterizes the quantum state of the system.

Let us start with formulating the known criterion for nonclassical states that is based on the P distribution. The Glauber-Sudarshan representation of the density operator [11],

$$\hat{\rho} = \int d^2\alpha \, P(\alpha) \, |\alpha\rangle\langle\alpha| \,, \tag{2}$$

$|\alpha\rangle$ being a coherent state, allows one to express normally ordered moments of annihilation and creation operators in close analogy to classical mean values,

$$\langle \hat{a}^{\dagger n} \hat{a}^m \rangle = \int d^2\alpha \, P(\alpha) \, (\alpha^*)^n \alpha^m . \tag{3}$$

Consequently, expectation values of normally ordered observables correspond to the classical expressions, if the following requirement is fulfilled [8–10]: the quasiprobability $P(\alpha)$ must be well behaved in the sense of a classical probability measure,

$$P(\alpha) \equiv P_{\mathrm{cl}}(\alpha) \,. \tag{4}$$

That is, it should be possible to construct a classical probability measure that agrees with the quasidistribution $P(\alpha)$.

A quantum state that fulfills this condition will be called in the following a quantum state having a classical counterpart. This does not mean that it really behaves like a classical one for the following reason. The definition of the classical counterpart relies on normally ordered observables. Replacing an operator $\hat{O} = \hat{O}(\hat{a}^\dagger, \hat{a})$ with its normally ordered form, $:\hat{O}(\hat{a}^\dagger, \hat{a}):$, corresponds to a "subtraction" of the ground-state noise effects [16]. Various observables of such a state, however, may exhibit ground-state noise effects. Consequently, a state having a classical counterpart may be said to behave like a classical one if the noise effects of the oscillator ground state are of minor importance for describing its properties.

For a quantum state to be nonclassical, one may formulate the following conditions: (a) The ground-state noise effects play a significant role for its characterization. (b) The P function fails to be interpreted as a classical probability measure,

$$P(\alpha) \neq P_{\mathrm{cl}}(\alpha) \,. \tag{5}$$

If one of these conditions is fulfilled the state may be called nonclassical. The condition (a), which implies that operator-ordering prescriptions are relevant for characterizing the given state, is of particular importance for small average occupation numbers of the oscillator. In this sense it closely corresponds to the first condition formulated by Mandel [9]. In its present form, however, it is more directly connected with condition (b) since the ground-state noise effects and the ordering prescription reflected by the P function are closely related to each other.

The advantage of the criterion for nonclassical states based on the P representation consists of the fact that it is based on the full information on the quantum state under study. Thus it applies to all observable properties of that state. On the other hand, a serious disadvantage of this criterion consists of the fact that it is hardly used for interpreting experimental results. For various nonclassical states the quasidistribution $P(\alpha)$ is not only negative but highly singular; thus it is far from being determined by measurements.

To overcome this problem we will reformulate the condition (b) by using the observable $\hat{x}(\varphi)$ defined in Eq. (1) with the aim to obtain an observable criterion for a nonclassical state. The probability distribution $p(x,\varphi)$ for observing the value x for the (arbitrary but fixed) phase φ can be expressed in terms of the characteristic function $G(k,\varphi)$,

$$p(x,\varphi) = \frac{1}{2\pi} \int dk \, e^{-ikx} G(k,\varphi) \,, \tag{6}$$

where

$$G(k,\varphi) = \langle e^{ik\hat{x}(\varphi)} \rangle \,. \tag{7}$$

For relating this probability distribution to the situation in classical physics, we "subtract" the ground-state noise effects by introducing a noise-subtracted version of the distribution, $\tilde{p}(x,\varphi)$. It is obtained from Eq. (6) by replacing therein the characteristic function $G(k,\varphi)$ with its normally ordered version $\tilde{G}(k,\varphi)$,

$$\tilde{G}(k,\varphi) = \langle :e^{ik\hat{x}(\varphi)}: \rangle \,. \tag{8}$$

Based on the explicit form of the operator $\hat{x}(\varphi)$ and on the Baker-Campbell-Hausdorff formula, it is easy to derive the relation

$$G(k,\varphi) = \tilde{G}(k,\varphi) e^{-k^2/2} . \tag{9}$$

The second factor represents the (phase-insensitive) characteristic function of the ground state of the oscillator. Inserting Eq. (9) into (6) yields the physical quadrature distribution in terms of a convolution,

$$p(x,\varphi) = \int dx' \, \tilde{p}(x',\varphi) p_{\mathrm{gr}}(x - x') \,, \tag{10}$$

of the noise-subtracted distribution with the (phase-insensitive) distribution of the ground state, $p_{\mathrm{gr}}(x) = \exp(-x^2/2)/\sqrt{(2\pi)}$.

For relating the properties of the P function to those of the distribution $\tilde{p}(x,\varphi)$ it is advantageous to express the latter by using the P representation of the density operator given in Eq. (2). From Eqs. (6) and (8), after performing the k integration, we derive

$$\tilde{p}(x,\varphi) = \int d^2\alpha\, P(\alpha)\delta[x - x_\alpha(\varphi)], \tag{11}$$

where $x_\alpha(\varphi) = \alpha e^{i\varphi} + \alpha^* e^{-i\varphi}$. This result formally corresponds to the expression of the distribution $\tilde{p}(x,\varphi)$ in terms of a classical stochastic process (see, e.g., [17]). It has the properties of a classical stochastic process, $\tilde{p}(x,\varphi) \equiv p_{\mathrm{cl}}(x,\varphi)$, provided that $P(\alpha)$ displays the properties of a probability measure [cf. Eq. (4)].

This allows us to reformulate the criterion (b) for a nonclassical state as follows: The noise-subtracted quadrature distribution, $\tilde{p}(x,\varphi)$, fails to be interpreted as a classical stochastic process, $\tilde{p}(x,\varphi) \neq p_{\mathrm{cl}}(x,\varphi)$. This is an important result: Well-behaved P distributions are directly related to well-behaved noise-subtracted quadrature distributions. Consequently, a quantum state has nonclassical properties if its noise-subtracted quadrature distribution does not exhibit the properties of a classical stochastic process.

So far, ill-behaved P functions will also lead to ill-behaved distributions $\tilde{p}(x,\varphi)$, so that we still have not obtained a measurable criterion. We have to derive an observable condition that is necessary and sufficient for $\tilde{p}(x,\varphi)$ not to be a probability measure or, equivalently, for $\tilde{G}(k,\varphi)$ not to be a classical characteristic function (CCF). For doing this, we start from the theorem of Bochner [18]: "a continuous function $G(k)$, obeying the condition $G(0) = 1$, is a CCF if and only if it is positive definite." Here positive definiteness means that, for arbitrary real numbers k_i $(i = 1,\ldots,n)$, arbitrary complex numbers ξ_i, and any integer n, the condition

$$S \equiv \sum_{i,j=1}^{n} G(k_i - k_j)\xi_j^*\xi_i \geq 0 \tag{12}$$

is fulfilled. Consequently, the existence of nonclassical properties requires that there exist values of k and φ for which the normally ordered characteristic function, $\tilde{G}(k,\varphi)$, fails to be positive definite. The characteristic function G has the properties of a classical characteristic function since it is the characteristic function for the quadrature distribution, which is clearly a probability distribution, so that Eq. (12) is valid. Nonclassicality implies that the characteristic function $\tilde{G}$ of the noise-subtracted quadrature distribution violates Eq. (12), so that the inequality

$$\tilde{S} < 0 \leq S \tag{13}$$

must be fulfilled [$\tilde{S}$ is defined by Eq. (12) with $\tilde{G}$ in place of G]. In this inequality we express $\tilde{G}$ by use of Eq. (9) in terms of G. Making use of the fact that the latter is a CCF and thus fulfills the condition $|G| \leq 1$, one can eventually derive the necessary and sufficient condition for the validity of the inequality (13) by estimation of the negative contributions therein. This yields the fact that the function $\tilde{G}$ violates Eq. (12) if and only if there exist values of k and φ for which the condition

$$|\tilde{G}(k,\varphi)| > 1 \tag{14}$$

is fulfilled. Because of Eq. (9), the characteristic function $G(k,\varphi)$ of the measurable quadrature distribution as a function of k decays more slowly than the characteristic function of the oscillator ground state,

$$|G(k,\varphi)| > e^{-k^2/2}. \tag{15}$$

In the Fourier domain a slow decay corresponds to narrow structures in the distribution.

In place of (b) we can formulate the following necessary and sufficient condition for a state to be a nonclassical one in terms of measurable quantities: (b*) There exist values of the phase φ for which the quadrature distribution $p(x,\varphi)$ exhibits structures that are narrower than the distribution of the ground state [19]. Correspondingly, for those φ values the characteristic function $G(k,\varphi)$ as a function of k decays more slowly than the characteristic function of the ground state [cf. Eq. (15)].

The criterion (b*) is indeed fulfilled for typical nonclassical states, such as Fock states, squeezed states, coherent superpositions of coherent states, and others. Concerning the measurability of both the quadrature distribution and its characteristic function there exists an extensive literature including light and matter systems (for a recent review see [15]).

From Bochner's theorem the criterion (b*) could be derived independently of the one using the P function. It remains to demonstrate that the criterion (b*), which is equivalent to the conditions (14) or (15), exactly agrees with the original criterion (b). In terms of characteristic functions Eq. (11) reads as

$$\tilde{G}(k,\varphi) = \Phi(ike^{-i\varphi}), \tag{16}$$

where $\Phi(\alpha) = \int d^2\beta \exp(\alpha\beta^* - \alpha^*\beta)P(\beta)$ is the characteristic function of the P function. From the equality (16) of the two characteristic functions it follows that there is a one-to-one correspondence between the positive definiteness of $\Phi(\alpha)$ and $\tilde{G}$. Consequently, the P function fails to be a probability measure according to condition (b) if and only if the measurable criterion (b*) is fulfilled.

Introducing the s-parametrized quasidistributions $P(\alpha;s)$ of Cahill and Glauber [20] for $s < 1$, the corresponding characteristic functions are given by

$$\Phi(\alpha;s) = \Phi(\alpha)e^{-(1-s)|\alpha|^2/2}, \tag{17}$$

where the exponential factor is the characteristic function of the ground state. Combining this relation with Eqs. (14) and (16), we arrive at the following criterion for a nonclassical state in terms of quasidistributions: (b**) There exist structures in the phase-space distribution $P(\alpha;s)$ that are narrower than the ground-state distribution $P_{\mathrm{gr}}(\alpha;s)$. Equivalently, the decay of the related characteristic function may survive the decay of the characteristic function $\Phi_{\mathrm{gr}}(\alpha;s)$.

The condition (b**) is completely equivalent to the condition (b) we were starting from. To prove this, we may again use the Bochner theorem [18]. We apply Eq. (17) for $s = -1$ together with the fact that the characteristic function $\Phi(\alpha; -1)$ of the Husimi Q function fulfills the properties of a CCF. In the same way as proving that Eq. (14) is a sufficient and necessary condition to obey the condition (13), one concludes that (b**) is a necessary and sufficient condition for the P function not to be a probability measure, Eq. (5).

Let us briefly comment on the application of the necessary and sufficient criteria (b*) and (b**) for the case of realistic detection with nonunity quantum efficiency. In the case of balanced homodyning, one measures a convolution of the quadrature distribution with a Gaussian noise distribution [21]; the width of the latter increases with decreasing quantum efficiency η of the detection system. Because of the Gaussian nature of the added noise it is straightforward to prove that the necessary and sufficient condition (b*) also remains valid for the distribution $p(x, \varphi; \eta)$ that is measured: it shows structures narrower than those of the (measured) ground-state distribution $p_{\mathrm{gr}}(x, \varphi; \eta)$. The situation for the criterion (b**) is similar: in the unbalanced homodyne measurement of the quasiprobabilities $P(\alpha; s)$ the quantum efficiency may be included in the value of s [22], so that the criterion also remains valid for realistic detection. Needless to say, for small quantum efficiencies the nonclassical effects are more and more smoothed out and may practically become unobservable.

In conclusion, we have derived criteria for nonclassical states that are based on measurable distributions such as quadrature or phase-space distributions. A quantum state has no classical counterpart when these functions show structures that are narrower than the corresponding distributions of the ground state of the oscillator. Equivalently, in such cases the characteristic functions of the quantum state exhibit a slower decay behavior than those for the ground state. These criteria are necessary and sufficient for the failure of interpreting the P function as a probability measure, which is the most commonly accepted but nonobservable criterion for a nonclassical state.

This work was supported by the Deutsche Forschungsgemeinschaft. The author gratefully acknowledges valuable discussions with D. Elstner, Z. Kis, and L. Mandel.

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