Discrete Coherent Amplification of Oscillations by Nonresonant Forcing

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A new mechanism of generation of oscillations in a linear forced oscillatory system is found. Natural oscillations may be generated at a "sharp" pulse (rapid variation) of the natural frequency. In this process oscillations are generated by nonresonant forcing, e.g., by the action of a constant, nonperiodic or periodic force (with driving frequency much less than the natural one). Repetitive pulses of the natural frequency result in emergence of oscillations that interfere and may give a powerful resultant output. These phenomena relate to a basis of the theory of open linear oscillatory systems.

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A variety of phenomena of a highly different nature (classical and quantum mechanical, electromagnetic, chemical, biological, etc.) exhibit oscillatory behavior. Undoubtedly, any novelty in the theory of oscillations will have various applications in many branches of the natural sciences. This becomes increasingly true if a novelty relates to the class of open linear oscillatory systems and has to do with the basis of the theory. This is the case for the topic of the present Letter. We present a new mechanism of generation of the natural oscillations in a forced linear oscillatory system and demonstrate one of the possible outcomes of this phenomenon—a discrete coherent amplification of oscillations.

Phenomena of the generation of natural oscillations in open linear oscillatory systems may be tentatively classified into three essentially different types. The simplest of these is the resonant phenomenon—resonance of the periodic external force and the natural oscillations of the system with constant natural frequency (see, e.g., Ref. [1]). Also well understood is the generation of oscillations due to the parametric resonance. In this case the open character of the system is reduced to the periodic variation of the natural frequency itself (see, e.g., Refs. [1,2]). The phenomenon of the third type is the nonresonant (impulsive) generation of oscillations caused by a pulse in the external force (see, e.g., Refs. [3,4]). A new and in this sense a fourth type of generation phenomenon is described in the present Letter. To get insight into the nature of this phenomenon we first recapitulate the nonresonant phenomenon introduced in Ref. [5].

The phenomenon of an abrupt emergence of waves from vortices has been revealed in the study of linear dynamics of perturbations in a simplest hydrodynamical shear flow—unbounded parallel compressible flow with uniform shear of velocity—which allows for only one mode of a wavelike solution: acoustic waves. The dynamics of perturbations in this flow has been studied in the framework of the nonmodal approach. Originated by Lord Kelvin in 1887 (see Ref. [6]) this approach has become extensively used in hydrodynamics since the 1990's (see Refs. [7–11]). Under this approach the linear dynamics of perturbations in the flow may be described in terms of a linear forced oscillatory system:

$$\ddot{\Psi}(t) + \omega^2(t)\Psi(t) = f(t), \tag{1}$$

where $\omega(t)$ and $f(t)$ are the time dependent natural frequency and external force, respectively. Mean flow velocity shear results in the following time dependence of the system parameters: $\omega^2(t) \propto [1 + (t - t^*)^2]$, $f(t) \propto (t - t^*)$, and $\dot{f}(t)/f(t) \ll \omega(t)$, i.e., the natural frequency has a minimum at $t = t^*$ and the external force is a slowly varying, nonperiodic function of time. Equation (1) describes two different modes of perturbations: (1) acoustic wave mode (natural oscillations of the system)—$\phi(t)$ that is described by the general solution of the corresponding homogeneous equation; (2) vortex/aperiodic mode—$\psi(t)$ that is driven by $f(t)$ and is associated with the particular solution of the equation, i.e., originates from the equation inhomogeneity. In a general case the complete solution of Eq. (1) takes the following form: $\Psi(t) = \phi(t) + \psi(t)$. It has been shown numerically that if the aperiodic solution of the system is present the fast variation of the natural frequency $[|\dot{\omega}(t)|/\omega^2(t) > 0.2]$ results in an abrupt emergence of natural oscillations at the point in time when the natural frequency passes through its minimal value (see Ref. [5]). Analogous phenomena have been found in magnetohydrodynamic as well as in plasma shear flows [12,13]. The discussed phenomenon may be comprehended as a "birth" of natural oscillations. Importantly, the outlined phenomenon is described by a quite general model equation of an open linear oscillatory system [see Eq. (1)] that is externally affected in two ways: (i) explicitly, by a slowly varying driving force; (ii) parametrically, by a temporal variation of the natural frequency.

The aim of this Letter is to describe a model of a linear forced oscillatory system that allows for a new type of the generation phenomenon that we have called the oscillation birth phenomenon, to get insight into the nature of this phenomenon, and to study the effect of repetitive pulses of the natural frequency in such a system. For these purposes numerical as well as qualitative perturbative analysis is used. We consider open linear oscillatory systems [see Eq. (1)] with pulsing natural frequencies, i.e., with frequencies that rapidly vary during a limited time interval. Therewith the external force is constant or slowly varying in time: $\dot{f}(t)/f(t) \ll \omega(t)$. Further in our exploration we consider that only the particular (aperiodic) solution is excited initially: $\Psi(0) = \psi(0)$, $\dot{\Psi}(0) = \dot{\psi}(0)$.

First, we present the results of the numerical calculations. The dynamics of the model oscillatory system with a single pulse in the natural frequency and a constant external force is shown in Fig. 1: $f(t) = 1$ and $\omega^2(t) = 0.5 + 0.2/[1 + (t - t^*)^2]$. The initial value of the particular solution is chosen using the following approximate form: $\psi(0) = 1/\omega^2(0)$, $\dot{\psi}(0) = 0$. The accuracy of this choice is defined by the small parameter $|\dot{\omega}(0)|/\omega^2(0)$ but may be increased using a numerical iterative method. The numerical results show the generation of natural oscillations at the natural frequency pulse, at $t = t^*$. The further dynamics of the generated portion of oscillations $[\phi(t)]$ is independent from the forcing and may be described by the equation of free oscillations—the homogeneous part of Eq. (1).

Similar results are obtained for periodic forcing $[f(t) \propto \cos(\Omega_{\rm ex} t)]$, when the frequency of the driving force is much smaller than the natural one $[\Omega_{\rm ex} \ll \omega(t)]$. The calculations also show that the amplitude of the emerged oscillations increases with growth of the "sharpness" of the effective pulse in the function $f(t)/\omega^2(t)$.

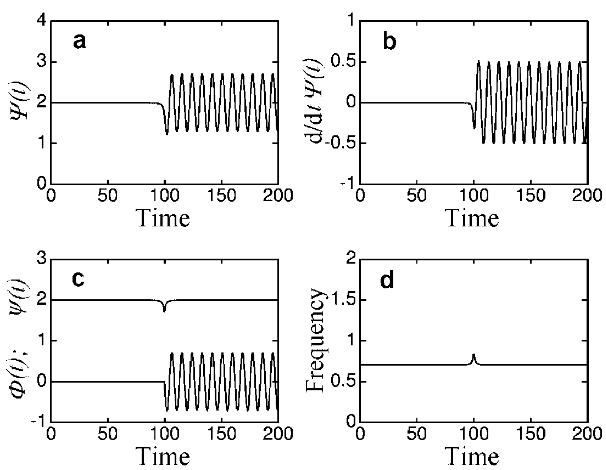

FIG. 1. The solution of Eq. (1) $\Psi(t)$ and its first derivative $\dot{\Psi}(t)$ are presented on graphs (a) and (b), respectively. Graph (d) shows that the natural frequency of the system undergoes a pulse type variation at $t = t^* = 100$. Graph (c) shows the aperiodic [$\psi(t)$, top graph] and the oscillating [$\phi(t)$, bottom graph] components of the solution individually. The oscillatory and aperiodic solutions are split using the symmetry properties of these solutions (see Ref. [5]). The abrupt emergence of the oscillations is traced from the dynamics of $\phi(t)$.

To interpret the physics of the described phenomenon we carry out a perturbative analysis of Eq. (1). We consider a sample case of a linear forced oscillatory system. The external force is constant and the natural frequency is constant apart from the pulse type variation in the vicinity of $t = t^*$ with duration Δt. Formally

$$f(t) = f_0 = \text{const},$$

$$\omega^2(t) = \omega_0^2[1 + \epsilon a(t)], \qquad \epsilon \ll 1\,, \tag{2}$$

$$\left\{ \begin{array}{ll} a(t) = 0, & \text{when } |t - t^*| > \Delta t/2, \\ \epsilon|\dot{a}(t)| \ll \omega_0, & \text{when } |t - t^*| < \Delta t/2, \end{array} \right\}$$

where ϵ is a parameter that is small enough to ensure the applicability of the perturbative analysis, and $a(t)$ is the continuous function characterizing the temporal variation of the natural frequency. We expand the full solution $\Psi(t)$ and its general $[\phi(t)]$ and particular $[\psi(t)]$ components in ϵ,

$$\Psi(t) = \Psi_0(t) + \epsilon\Psi_1(t) + \cdots. \tag{3}$$

Retaining the zero and first terms in ϵ, Eqs. (1), (2), and (3) yield

$$\ddot{\Psi}_0(t) + \omega_0^2\Psi_0(t) = f_0\,, \tag{4}$$

$$\ddot{\Psi}_1(t) + \omega_0^2\Psi_1(t) = -\omega_0^2 a(t)\Psi_0(t)\,. \tag{5}$$

Consider that initially, before the pulse of the natural frequency [when $t < t^* - \Delta t/2$ and $a(t) = 0$], the natural oscillations are absent and internal restoring and external forces compensate each other:

$$\begin{array}{ll} \phi(t) = \phi_0 = 0 & \text{when } t < t^* - \Delta t/2\,, \\ \Psi_0 = \psi_0 = f_0/\omega_0^2, & \ddot{\psi}_0 = 0\,. \end{array} \tag{6}$$

The pulse of the natural frequency results in a corresponding pulse of the source term in Eq. (5): $-\omega_0^2 a(t)\psi_0 = -f_0 a(t)$. Hence, under some circumstances the natural frequency variation may result in the generation of natural oscillations if the particular solution is present ($\psi_0 \neq 0$). Conditions for the effective generation of the oscillations may be obtained using the Fourier expansion of quantities in Eq. (5):

$$\left\{ \begin{array}{c} \tilde{\Psi}_1(\sigma) \\ \tilde{a}(\sigma) \end{array} \right\} = \int_{-\infty}^{\infty} dt \left\{ \begin{array}{c} \Psi_1(t) \\ a(t) \end{array} \right\} \exp(-i\sigma t)\,. \tag{7}$$

Hence, Eqs. (5), (6), and (7) yield

$$\tilde{\Psi}_1(\sigma) = \frac{f_0\tilde{a}(\sigma)}{(\sigma^2 - \omega_0^2)}\,. \tag{8}$$

Generally, the resonance needs $\tilde{a}(\omega_0) \neq 0$. Moreover, an efficient resonance needs a noticeable value of $\tilde{a}(\omega_0)$. Assuming a simple form of the function $a(t)$ and the increase of $|\tilde{a}(\sigma)|$ with a decrease of $|\sigma|$ we obtain the condition necessary for an effective generation of natural oscillations: $\Delta\sigma \gg \omega_0$, where $\Delta\sigma$ is the spectral width of the frequency pulse. Using the estimation of the pulse duration $\Delta t \simeq 1/\Delta\sigma$, we reach the following condition:

$$\omega_0 \Delta t \ll 1 . \tag{9}$$

In other words, the generation of natural oscillations may occur at a sharp pulse of the natural frequency, when the period of the oscillations substantially exceeds the duration of the natural frequency pulse itself. This estimate of the natural frequency pulse has been confirmed using the numerical calculations. Generally, numerical calculations show that the amplitude of the excited natural oscillations grows with the sharpness of the natural frequency pulse. (However, the above qualitative perturbative analysis is applicable only in particular cases when the rapidly varying part of the natural frequency is not large.) Notwithstanding the fact that Eqs. (4)–(9) are obtained for a constant external force, they also well approximate the case of a slowly varying external force ($\Omega_{\mathrm{ex}} \ll \omega_0$) well.

What will be the effect of repetitive pulses of the natural frequency in a linear forced oscillatory system?

Obviously, oscillations will be excited at every sufficiently "sharp" pulse of the natural frequency. The emerged portions of oscillations that are independent from further forcing will linearly interfere. The amplitude of the generated natural oscillations of the system will be defined by the interference of the previously emerged oscillations. Hence, every subsequent pulse of the natural frequency will increase or decrease the amplitude of the natural oscillations depending on the phase difference between the previously generated and the emerged portion of oscillations. Suppose, that regular repetitive pulses of the natural frequency excite coherent portions of oscillations. In this case oscillations that emerged at the different pulses are in phase. Interference of such portions will result in a simple sum of their amplitudes. Hence, the amplitude of the natural oscillations will grow proportionally to the number of pulses ($\phi_n = n\phi_0$) and the energy—proportionally to the squared number of pulses ($E_n = n^2 E_0$). Thus, regular repetitive pulses of the natural frequency may lead to a powerful process of stepwise, intrinsically discrete amplification of the natural oscillations by nonresonant forcing. In this sense, the presented amplification process is clearly distinguishable from another phenomenon caused by a temporal variation of the natural frequency: the parametric resonance. The amplification of oscillations due to the parametric resonance is continuous and proceeds smoothly in time, while the discrete coherent amplification proceeds demonstratively stepwise.

Further we use numerical analysis to confirm the above qualitative results. We consider linear oscillatory systems with a natural frequency having regular repetitive pulses. For simplicity we use the periodic natural frequency modeled by the following equation: $\omega^2(t) = 0.5\{1 + e/[e + 1 + \cos(2\pi t/T_i)]\}$. It undergoes a rapid variations repeating in every time interval T_i. The sharpness of the frequency pulses is defined by the parameter e. Coherent generation of oscillations is reached by selecting the values of the parameter $\omega_0 T_i$. Generated oscillations are in phase when this parameter is a multiple of 2π. In all cases this parameter is taken high enough to exclude the parametric resonance: $\omega_0 T_i \gg 1$. The cases of the constant [$f(t) = f_0$] and periodic external force [$f(t) = f_0 \cos(\Omega_{\mathrm{ex}} t)$] are presented in Figs. 2 and 3, respectively. In the latter case the frequency of the external force is taken to be much less than the natural one [$\Omega_{\mathrm{ex}} \ll \omega(t)$]. Discrete coherent amplification of the natural oscillations has clearly occurred in both cases (see Figs. 2 and 3). The total energy of natural oscillations is calculated using the following approximate

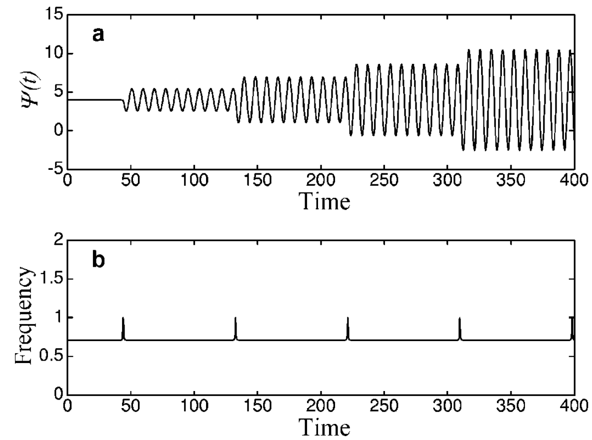

FIG. 2. Graph (a) presents the solution of Eq. (1) $\Psi(t)$ at the constant external force. The natural frequency of the system is presented on graph (b). Here $f_0 = 2$, $T_i = 88.5$, and $e = 0.0001$. The coherent excitation of oscillations is explicitly seen from (a) and is provided by the coherence parameter $\omega_0 T_i = 20\pi$.

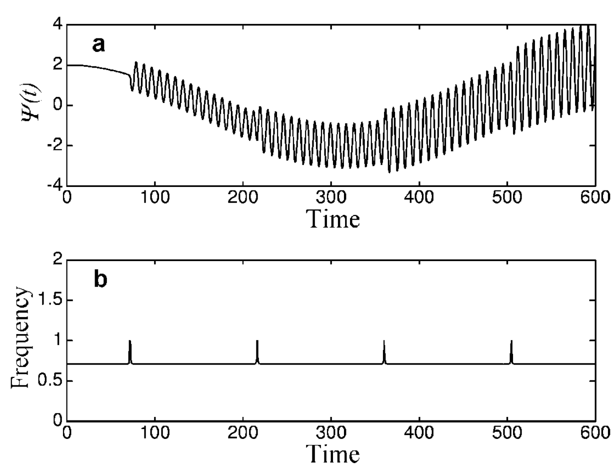

FIG. 3. Graph (a) presents the solution of Eq. (1) $\Psi(t)$ at the periodic external force. The natural frequency of the system is presented on graph (b). Here $f_0 = 1$, $\Omega_{\mathrm{ex}} = 0.01$, $T_i = 114.11$, and $e = 0.0001$. Different from Fig. 2, the amplification of the natural oscillations occurs on the background of the adiabatic response to the periodic external force.

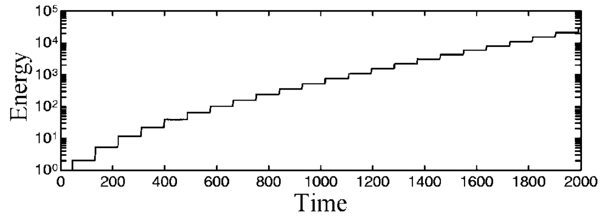

FIG. 4. The total energy of natural oscillations vs time at the same parameters as in Fig. 2 is shown for the longer time period. The total energy of oscillations grows stepwise, proportionally to the square of the pulse number.

form: $E(t) = [\dot{\phi}(t)]^2 + \omega^2(t)\phi^2(t)$. The power of the amplification process is demonstrated in Fig. 4.

The presented phenomenon of the generation of natural oscillations principally differs from the classical resonant one. It is also different from the nonresonant phenomenon caused by the "push"—the pulse of the external force. In the latter case the oscillations are fed directly from a pulsing external force, i.e., the energy of the generated oscillations is determined by the pulse source power. With regard to the presented phenomenon, it is initiated only by the pulse source: Initially, there is a tense balance between the internal restoring and external forces. The natural frequency pulse breaks this balance and triggers the excitation process by activating the internal restoring and external forces of the system. Specific to the phenomenon is that the natural frequency pulse needs much less energy than the pumping work done in the system to generate oscillations. Actually, it initiates the release of the stress energy that is created by the external forces. Generally speaking, it appears that the parametrical pulse type effect on the oscillatory system under some circumstances should result in a much more powerful response than the explicit effect of the analogous external force.

Finally we shortly summarize the presented study. We have found a new type of generation of natural oscillations in open linear oscillatory systems. Emergence of the natural oscillations (birth of the oscillations) occurs in oscillatory systems that are externally affected in two ways: explicitly, by a constant or slowly varying driving force, and parametrically, by a rapid variation (sharp pulse) of the natural frequency when the period of the natural oscillations well exceeds the duration of the pulse itself. The natural frequency pulse breaks the system balance and triggers the excitation of natural oscillations that are pumped directly from the nonresonant driving force. Regular repetitive pulses of the natural frequency of the forced oscillatory system may lead to a powerful resultant response at some parameters of the system. This amplification process is a result of the discrete coherent summing of the excited portion of oscillations. Therewith, the frequency of the generated oscillations may be qualitatively higher than the frequency of the initiating pulsing source. Characteristic features of the presented amplification process show that it may be easily realized in laboratory experiments for a wide spectra of oscillations. The general nature of the presented phenomenon ensures its essential, and, at times, exotic manifestations in nature.

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