

## Characteristics of Two-Dimensional Turbulence That Self-Organizes into Vortex Crystals

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Experiments have found that freely relaxing turbulence in inviscid, incompressible two-dimensional Euler flows can self-organize into ordered structures—vortex crystals—in which a number  $N_c \sim 2-20$  of strong vortices form stable, rigidly rotating patterns in a low vorticity background. In this paper we show that  $N_c$  can be roughly predicted from properties of the flows in the early stage of the turbulent relaxation.

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Turbulence in inviscid, incompressible, two-dimensional (2D) fluids is applicable to large scale geophysical and astrophysical flows. These fluids evolve according to the 2D Euler equations:  $\partial_t \omega + \mathbf{v} \cdot \nabla \omega = 0$ ,  $\omega = \hat{\mathbf{z}} \cdot \nabla \times \mathbf{v}$ , where  $\mathbf{v}(\mathbf{r}, t)$  and  $\omega(\mathbf{r}, t)$  are the velocity and vorticity fields of the flow, respectively, and  $\hat{\mathbf{z}}$  is the unit vector perpendicular to the plane of the flow. The velocity and the vorticity are related via the stream function  $\psi(\mathbf{r}, t)$ :  $\mathbf{v} = \nabla \times \psi \hat{\mathbf{z}}$ ,  $\nabla^2 \psi = -\omega$ .

Over the years, numerical simulations and experiments have found that, from a wide variety of unstable initial conditions, 2D Euler flows quickly organize into large numbers of strong vortices (intense patches of vorticity) and a filamentary, low vorticity background. Subsequently, the turbulence is dominated by the chaotic mutual advection of the strong vortices and mergers of like sign strong vortices. The mergers usually continue until only a single strong vortex or a pair of opposite signed strong vortices remains [1,2].

However, experiments have discovered that 2–20 strong vortices can sometimes remain in the final relaxed state. The vortices settle down to a rigidly rotating equilibrium pattern in a low vorticity background. The pattern lasts about  $10^4$  rotations of the flow, until the strong vortices are dissipated by nonideal effects. This equilibrium state is called a vortex crystal [3,4].

Experimentally, the formation of the vortex crystals depends on delicate control of the initial vorticity distribution of the flow. Slight variations of the initial condition can result in vortex crystals with different numbers of strong vortices, and from some initial conditions, no vortex crystal forms.

In this paper we show that the formation of vortex crystals and the number  $N_c$  of the strong vortices remaining in them can be roughly predicted from characteristics of the turbulent flows in their early stages of evolution. We focus on flows with a single sign of vorticity confined within a free-slip circular boundary.

Our analysis relies on the following physical picture, supported by recent vortex-in-cell simulations [5] and analytic theory [6]. Vortex crystals form because of the interaction between the strong vortices and the low vorticity background. While advecting chaotically and

merging occasionally with each other, the strong vortices ergodically mix the background, causing the fluid entropy of the coarse-grained background to increase. The mixing of the background, in return, “cools” the chaotic motions of the strong vortices and leads to the formation of vortex crystal states. The physics of cooling is similar in many respects to that of a marble rolling across the floor: the marble slows to a stop because the entropy of the floor is increased, implying an irreversible flow of energy from the marble to the floor.

This physical picture suggests that two time scales are important in vortex crystal formation: the average time  $\tau_m$  between merger events, and the time  $\tau_c$  required to cool the chaotic motions of the strong vortices through their interaction with the background. When  $\tau_m$  becomes longer than  $\tau_c$ , the strong vortices cool into a crystal state before the next merger event can occur and the mergers cease, leaving  $N_c$  vortices in the final state.

We can estimate  $\tau_m$  from the observed time evolution  $N(t)$  of the number of the strong vortices in the early stage of the turbulent evolution. Numerical simulations [1] and experiments [2] have found that  $N(t)$  evolves according to a power law:

$$N(t) = N(t_0) (t/t_0)^{-\xi}, \quad (1)$$

where  $\xi > 0$  is a constant, and  $t_0$  is any time chosen from the range of times where the power law scaling is observed. Other quantities associated with the strong vortices also evolve in time according to power laws. For example, the average circulation per vortex of the strong vortices,  $\bar{\Gamma}(t)$ , increases in time as

$$\bar{\Gamma}(t) = \bar{\Gamma}(t_0) (t/t_0)^{\eta\xi}, \quad (2)$$

where  $\eta > 0$  is a constant.

There are theoretical arguments for the power law behavior of  $N(t)$  [7]. The punctuated scaling theory, which is based on a merger model that conserves the total energy and the maximum vorticity of the strong vortices, suggests that  $\xi = 0.70-0.75$ , and  $\eta = 0.5$  [8]. Although the theory is supported by some numerical calculations [9] and experiments done with a thin stratified layer of electrolyte [10], experiments that observed vortex crystals [3] have

shown that these exponents can take different values in different flow evolutions, ranging from  $0.2 < \xi < 0.7$  and  $0.2 < \eta < 0.9$ . In this Letter, we take Eqs. (1) and (2) as empirical laws, and measure the exponents from experiments and simulations.

The time scale  $\tau_m$  between mergers is given by the time required for the number of the strong vortices to decrease by one:  $\Delta N = -1$ . Then  $\Delta N/\tau_m = -1/\tau_m \approx dN(t)/dt = -\xi N(t_0)(t/t_0)^{-1-\xi}/t_0$ , where in the second step  $N \gg 1$  was assumed and in the last step Eq. (1) was used. Therefore,

$$\tau_m \approx -1/(dN/dt) = t_0 \xi^{-1} N(t_0)^{-1} (t/t_0)^{1+\xi}. \quad (3)$$

Equation (3) merely states that the mean time between mergers increases as the flow evolves, since the number of strong vortices decreases with time and close encounters between vortices become less likely.

To estimate the cooling time scale  $\tau_c$ , we recall that mixing of the background increases the fluid entropy of the coarse-grained background and drives the strong vortices towards equilibrium patterns. As the background is mixed and the strong vortices approach equilibrium positions, it becomes unlikely that a fluctuation will drive the strong vortices sufficiently close together to merge. Therefore, we estimate  $\tau_c$  as the time scale to mix the background.

Mixing in turbulent fluids is a complex problem. One simplification is to consider instead the mixing of passive scalars in a prescribed flow [11]. In our case, we assume that the strong vortices are the primary mixers of the background and study the chaotic advection of passive scalars in the fields of point vortices in order to understand the mixing of the background.

The velocity of a passive scalar in the flow field of  $N$  point vortices is given by  $dx/dt = \partial\psi_v(x, y, t)/\partial y$ ,  $dy/dt = -\partial\psi_v(x, y, t)/\partial x$ , where  $\psi_v(x, y, t)$  is the stream function due to the point vortices and depends on time due to the motion of the point vortices. Observe that  $\psi_v$  can be regarded as the Hamiltonian for a particle in one-dimensional motion, with  $x(t)$  and  $y(t)$  being the generalized coordinate and momentum.

If the point vortices form a rigidly rotating equilibrium,  $\psi_v$  is time independent in the rotating frame of the equilibrium. Then, the trajectory of the passive scalar is integrable and nonchaotic, and no mixing occurs. The trajectory can go around one of the point vortices, or a number of them, depending on the initial position. The initial positions for different types of trajectories are separated by separatrices. If the point vortices oscillate around their equilibrium positions, trajectories near the separatrices become chaotic. If the point vortices are completely out of equilibrium, their own motions are chaotic, and the trajectory of the passive scalar is also chaotic starting from all initial positions (except those very close to or very far from the strong vortices) [12]. We focus on this latter case here, since chaotic motion of the point vortices in their mutual fields (neglecting

the effect of the background on their motion) is a reasonable description of the early stages of the flow.

The trajectory of a passive scalar is chaotic if its Lyapunov exponent  $\lambda$  is positive, nonchaotic otherwise. The Lyapunov exponent of a trajectory is defined in terms of the difference  $\delta\mathbf{r}(t) = \mathbf{r}_1(t) - \mathbf{r}_2(t)$  of two infinitesimally close trajectories,  $\mathbf{r}_1(t)$  and  $\mathbf{r}_2(t)$ :  $\lambda = \lim_{t \rightarrow \infty} t^{-1} \ln[|\delta\mathbf{r}(t)|/|\delta\mathbf{r}(0)|]$ . We then identify the inverse cooling time  $\tau_c^{-1}$  with the average Lyapunov exponent  $\bar{\lambda}$  of a collection of passive scalars, counting only those passive scalars with positive  $\lambda$ .

We can estimate  $\bar{\lambda}$  with dimensional analysis. When the point vortices have approximately equal circulations and their motions are chaotic in a region of area  $A$ , the main physical quantities that determine  $\bar{\lambda}$  are the average circulation  $\bar{\Gamma}$  of the point vortices and the average distance  $D = \sqrt{A/N}$  between the nearby point vortices. This implies that

$$\tau_c^{-1} = \bar{\lambda} \approx \alpha \bar{\Gamma}/D^2 = \alpha \Gamma_T/A, \quad (4)$$

where  $\alpha$  is a constant, and  $\Gamma_T = N\bar{\Gamma}$  is the total circulation of the point vortices.

To check the validity of this dimensional analysis, we have calculated  $\bar{\lambda}$ , with the method proposed in Ref. [13], in the field of  $N$  identical point vortices, each with circulations  $4\pi$ . The point vortices are randomly placed initially in a circular region and are allowed to move according to their mutual interaction. We vary  $N$  from 5 to 50 and the area  $A$  of the circular region from  $0.2^2\pi$  to  $0.8^2\pi$ . 7000–8000 passive scalars are uniformly distributed initially in a circular region of radius 0.9. The result confirms Eq. (4), with  $\alpha \approx 0.031$  (see Fig. 1).

To show that  $\tau_c$  as given in Eq. (4) is indeed the time scale on which the mergers of the strong vortices tend to stop, we have performed several vortex-in-cell simulations, an example of which is shown in Fig. 2 [14]. In the simulation, five identical point vortices with total circulation  $\Gamma_T = 0.5$  are distributed within a low vorticity background generated by placing 500 vortices randomly inside a circular region of radius 0.4. These background vortices all

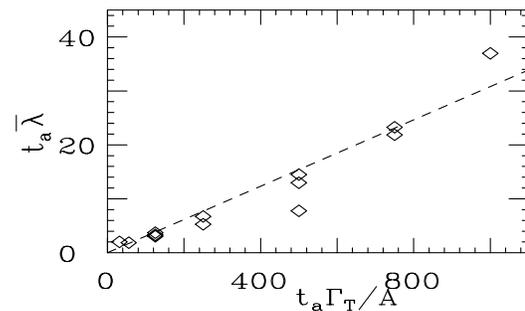


FIG. 1. Relation between  $\bar{\lambda}$  and  $\Gamma_T/A$ .  $t_a$  is an arbitrary time unit. The dashed line, with a slope of 0.031, is the best fit to the data.

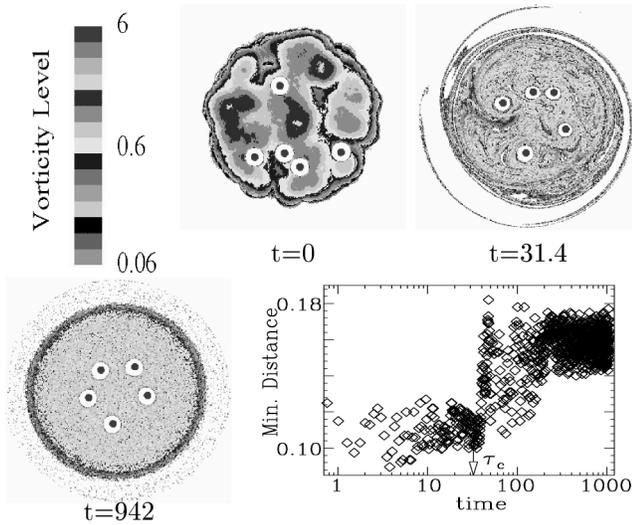


FIG. 2. A vortex-in-cell simulation with five point vortices in a random background. The flow evolution is shown at three times. The minimum distance between the point vortices is also plotted against time. The arrow in the figure indicates the cooling time  $\tau_c$  as evaluated by Eq. (4), with  $A = 0.4^2\pi$ .

have a radius of 0.08 and random maximum vorticity. The total circulation of the flow is 1. On a time scale of order  $\tau_c$ , these background vortices are sheared apart and mixed by the flow. This is seen in Fig. 2, where we plot the flow at three times. Cooling of the five point vortices is also demonstrated by the plot of the minimum distance between the point vortices. For the flow,  $\tau_c = 32.43$  as evaluated by Eq. (4), with  $A = 0.4^2\pi$ . This value is indicated with an arrow in the figure. After  $t = \tau_c$ , the lower limit for the minimum distances steadily increases, indicating a correspondence between the mixing of the background and cooling of the chaotic motions of the point vortices.

Other simulations have been run with circulation  $\Gamma_T$  of the five point vortices varying from 0.01 to 0.9. When  $\Gamma_T \ll 0.1$  or  $\Gamma_T \geq 0.9$ , Eq. (4) does not work. For  $\Gamma_T \geq 0.9$ , there is not enough background to cool the chaotic motions of the vortices, and for  $\Gamma_T \ll 0.1$ , the fluctuations in the background dominate the flow field. However, for  $0.1 \leq \Gamma_T \leq 0.8$ , Eq. (4) does appear to describe the cooling time of the point vortices with a similar accuracy to the example shown in Fig. 2.

The cooling time  $\tau_c$  depends on the total circulation of the strong vortices, which decreases as the strong vortices merge. From Eqs. (1) and (2), we obtain  $\Gamma_T(t) = N(t_0)\bar{\Gamma}(t_0)(t/t_0)^{\eta\xi-\xi}$ . Therefore, from Eq. (4) we obtain

$$\tau_c \approx \frac{A}{\alpha N(t_0)\bar{\Gamma}(t_0)} \left(\frac{t}{t_0}\right)^{\xi-\eta\xi}. \quad (5)$$

Comparison of Eqs. (3) and (5) shows that  $\tau_c$  grows in time more slowly than  $\tau_m$  since  $\eta > 0$ . Therefore, starting from  $\tau_m < \tau_c$ ,  $\tau_m$  will eventually catch up with  $\tau_c$  at  $t = t_1$ , and mergers of the strong vortices stop. Here  $t_1$  is found by setting  $\tau_c \approx \tau_m$ , and from Eqs. (3) and

(5) we arrive at  $t_1 \approx t_0[\xi A/\alpha t_0\bar{\Gamma}(t_0)]^{\frac{1}{1+\eta\xi}}$ . Accordingly, the number of the strong vortices in the vortex crystals is obtained by setting  $t = t_1$  in Eq. (1):

$$N_c \approx N(t_0) \left(\frac{\alpha t_0\bar{\Gamma}(t_0)}{\xi A}\right)^{\frac{\xi}{1+\eta\xi}}. \quad (6)$$

This equation shows that in order to form vortex crystals with many surviving strong vortices, initially the flow should have a large number of strong vortices with large average circulation, concentrated in a small area.

The prediction of Eq. (6) has been compared to both experiments with pure electron columns [3] and vortex-in-cell simulations. The exponents  $\xi$ ,  $\eta$ , as well as  $N$ ,  $\Gamma_v$ , and  $A$  are measured in the power law regime of the turbulent relaxation.  $A$  is determined by measuring the total area of flow with vorticity level larger than 1/4 of the average background vorticity. Then each flow in the power law regime predicts  $N_c$  according to Eq. (6).

In the simulations, initial conditions are generated by randomly distributing in a circular region a large number of vortices with equal radii and random maximum vorticity. Within one rotation time of the flow, a number of strong vortices and a low vorticity background form from this kind of initial distribution. Generally, the number of strong vortices formed increases with decreasing radius of the random vortices. In Fig. 3, we show a typical run of the simulation, starting with roughly 2000 random vortices with radius 0.005. For this small radius, several hundred strong vortices form initially and immediately begin to merge. The vorticity distributions at three times and the evolution of the number of the strong vortices are displayed. As expected from Eq. (6), a vortex crystal with a large number of strong vortices forms. Conversely, a simulation that starts with 1000 random vortices with radius

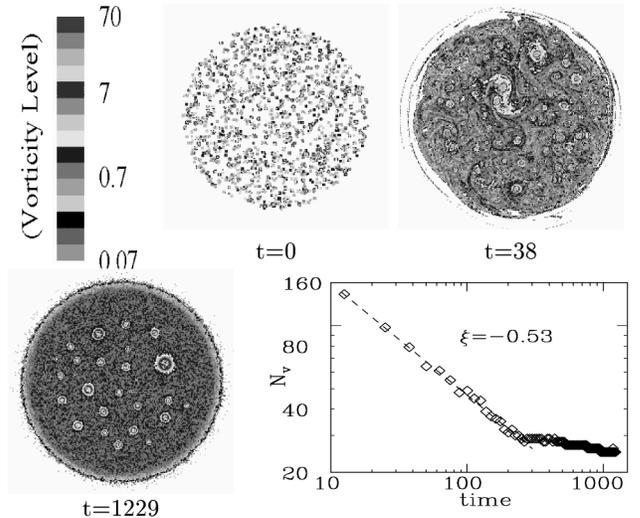


FIG. 3. Vortex-in-cell simulation starting with a random distribution of vortices with small radii. Images: Vorticity distributions at three different times. Plot: The evolution of the number of the strong vortices.

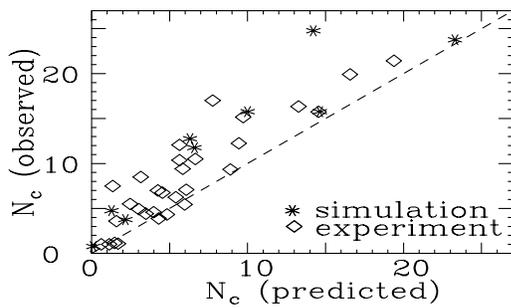


FIG. 4. Comparison of the predicted number  $N_c$  of the strong vortices in the vortex crystals with the  $N_c$  of the experiments and the simulations. Each data point represents a particular evolution of the turbulent flow.

0.02 forms only 36 strong vortices initially and results in only a single strong vortex in the final state, also as expected from Eq. (6).

In Fig. 4, we plot the predicted  $N_c$  from Eq. (6), averaged over the power law region of  $N(t)$ , against the observed  $N_c$  for the experiments and simulations. In the experiments the observed  $N_c$  fluctuates because of shot noise [3]. Therefore, we take the average value of the observed  $N_c$ . In the simulations, the number of strong vortices decreases slightly after the power law regime (see Fig. 3). This is due to the existence of a number of vortices with very small circulation compared to the average circulation of the strong vortices. These small vortices are essentially passive tracers and eventually merge into larger strong vortices. Since these small vortices are not part of vortex crystals, we take the observed  $N_c$  in the simulation at the final time after most small vortices have disappeared.

Figure 4 shows that the prediction of Eq. (6) clearly distinguishes the characteristics of flows that form vortex crystals with many strong vortices from those of flows that form no vortex crystals. However, the scatter in the data is quite large. This might be expected, given that the process of vortex crystal formation is chaotic.

Until now, vortex crystals have been observed only in turbulent flows with a single sign of vorticity, subject to a circular, free-slip boundary condition. It is of interest to determine whether vortex crystals can form in more general cases with both signs of vorticity and/or different boundary conditions. As we have shown in this paper, one requirement is that there should be many strong vortices in the initial stages of the turbulent flow. Our theory also

suggests that two conditions are crucial for vortex crystal formation. First, calculations similar to those we have done in Ref. [6] should be carried out for more general flows to reveal whether ordered, stable structures for the strong vortices can emerge by maximization of the fluid entropy of the low vorticity background. The second condition is that the mixing time scale  $\tau_c$  of the background must be sufficiently fast. This can be investigated by considering the chaotic advection of point vortices, as we have done in this paper. It is conceivable that  $\tau_c$  can vary considerably depending on the characteristics of the turbulent flow. For example, if there are approximately equal numbers of similar-sized positive and negative strong vortices, the mixing of the background may not be as efficient as the case we have studied in this paper, since the opposite signed strong vortices tend to form dipole pairs and hence at least partially cancel each other's mixing ability.

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