Vortex Entanglement in Disordered Superconductors

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Vortex entanglement and pinning in multiply connected disordered superconductors are studied. It is shown that the winding of vortices around repulsive obstacles is greatly enhanced by quenched columnar disorder and suppressed by point disorder, compared to the clean case. This leads to an additional contribution to the effective pinning force acting on vortices, which, unlike the conventional mechanisms of pinning, grows with temperature.

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The effects of interplay between thermal fluctuations and static disorder in the superconducting mixed state have been a subject of intensive studies in the past few decades. The surge of interest in this problem was prompted by the discovery of high-temperature superconductors (HTSC), whose peculiar material properties are such that the thermal fluctuations are capable of melting the Abrikosov vortex lattice in a significant part of the H-T phase diagram [1]. The matters are further complicated by a highly anisotropic crystalline structure of HTSC materials and the presence of quenched disorder of all sorts, giving rise to a rich phase diagram and non-trivial thermodynamic and transport properties (for a review, see Ref. [2]). Apart from a pure scientific interest, the studies of the mixed state properties of hard type-II superconductors (all HTSC materials fall into this category) are important from the viewpoint of technological applications, which usually require high critical currents and, therefore, large pinning forces.

Generally, the pinning of vortices at extended, i.e., linear or planar, defects (columnar pins, dislocations, twinning planes, or grain interfaces) is more effective than the collective pinning by pointlike impurities or vacancies. However, in all the cases, the strength of pinning rapidly diminishes as temperature increases. For instance, the localization length of a single flux line at an attractive columnar defect of radius r_0 and the binding energy U_0 grows exponentially with temperature T: $l_{\rm loc}(T) \simeq r_0 \exp[(T/T^*)^2]$, where $T^* \sim r_0\sqrt{U_0}$ [3]. As a result, at $T > T^*$ a flux line is virtually delocalized and can move freely. This leads to increasing the transverse flux flow resistivity in the mixed state, which is an obvious disadvantage for applications.

In this Letter, we study a different mechanism of pinning, related to the entanglement of vortices in superconductors with multiply connected geometry. The most distinct feature of this mechanism is that its strength *increases* with temperature. Qualitatively, if repulsive cylindrical obstacles are present in a superconductor, the flux lines may get entangled with them, just because of the thermal fluctuations of vortex positions in the plane perpendicular to the applied magnetic field. The entanglement prevents vortices from moving freely under the action of Lorentz force and gives rise to a significant "topological" contribution to the effective pinning force, which is determined, in a slab of thickness L, by the number of turns $n(L)$ of a flux line around an obstacle (the winding number). A similar mechanism works in flux liquids, where the mutual entanglement of vortices substantially increases pinning in the presence of a fairly small amount of disorder [4].

One possible experimental setup for studying the vortex entanglement in multiply connected superconductors was proposed by Nelson and Stern [5] (see Fig. 1). The density of flux lines trapped inside a bunch of densely packed columnar defects can be made much higher than outside. The vortex-filled tube thus presents a strongly repulsive obstacle for the rest of the vortices, which form a flux liquid. Another realization of repulsive obstacles is related to the artificial creation of a rod with higher critical temperature inside a superconductor, which would repel vortex cores.

The quantitative analysis of the vortex entanglement in the superconducting mixed state is based on the observation [2] that the flux lines can be thought of as the elastic strings stretched along the external magnetic field and subject to a pinning potential. If the displacement of a vortex line due to thermal fluctuations or disorder is smaller than the mean distance $n_v^{-1/2}$ between the vortices ($n_v = B/\Phi_0$ is the concentration of vortices, B is the magnetic induction inside the sample, and Φ_0 is the flux quantum), then the mutual entanglement can be neglected, and one deals with a single-vortex problem. The energy of a single flux line is given by

$$E[\mathbf{r}(z)] = \int_0^L dz \left\{ \frac{\epsilon_l}{2} \left(\frac{d\mathbf{r}}{dz} \right)^2 + U(\mathbf{r}(z), z) \right\}, \quad (1)$$

where $\epsilon_l = (\Phi_0 H_{c1})/4\pi$ is the line tension (H_{c1} is the lower critical field), and U is the random potential, describing the interaction with point or columnar disorder (in the latter case, U does not depend on z). The vortex partition function can be written as

$$Z(\mathbf{r}, \mathbf{r}; L) = \int \mathcal{D}\mathbf{r}(z) e^{-\beta E[\mathbf{r}(z)]} = \sum_{n=-\infty}^{\infty} Z_n(\mathbf{r}, \mathbf{r}; L), \quad (2)$$

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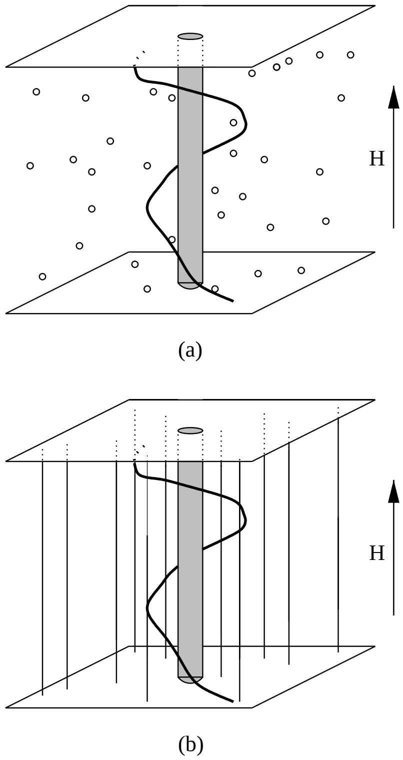

FIG. 1. Superconducting slab with a repulsive cylindrical obstacle of radius R_0 (shaded area), and point impurities (a) or columnar defects (b) in the bulk. An Abrikosov vortex (thick curve) is entangled with the obstacle.

where the path integral is taken over all trajectories $\mathbf{r}(z)$ such that $\mathbf{r}(0) = \mathbf{r}(L) = \mathbf{r}$, and Z_n is the constrained partition function of a vortex winding n times around the obstacle:

$$Z_n = \int \mathcal{D}\mathbf{r}(z)\delta\left(n - \frac{1}{2\pi}\int_0^L dz\,\frac{d\theta}{dz}\right)e^{-\beta E[\mathbf{r}(z)]}, \quad (3)$$

$\theta(z)$ is the angle between the radius-vector $\mathbf{r}(z)$ and some fixed direction in the transverse plane. The mean-square winding number $\langle n^2(L)\rangle = \int dn\, n^2 \mathcal{P}(n,L)$ is related to the normalized winding probability distribution $\mathcal{P}(n,L) = \langle \int d\mathbf{r}\, Z_n(\mathbf{r},\mathbf{r};L)\,[\int d\mathbf{r}\, Z(\mathbf{r},\mathbf{r};L)]^{-1}\rangle$, where the averaging over quenched disorder is implied.

Expressing the δ function in Eq. (3) as an integral over an auxiliary variable ϕ, and considering Eq. (1) as a Euclidean action in 2 + 1 dimensions, the vertical coordinate z playing the role of time [1], the path integral on the right-hand side of Eq. (3) is recognized as the Fourier transform $Z_n(\mathbf{r},\mathbf{r};L) = (1/2\pi)\int d\phi\, e^{-i\phi n}G_\phi(\mathbf{r},\mathbf{r};L)$ of the Green function of a 2D quantum particle with Hamiltonian

$$H = \frac{T}{2\epsilon_l}(-i\nabla - \mathbf{A})^2 + \frac{U(\mathbf{r},z)}{T}. \quad (4)$$

Here $A_\theta = \phi/2\pi r$ is the vector potential of a fictitious solenoid placed inside the obstacle and carrying magnetic flux ϕ. This way of imposing topological constraints dates back to the seminal works of Edwards [6]. In addition, the vortex partition function must satisfy the Dirichlet boundary condition at the surface of an impenetrable obstacle: $Z|_{r=R_0} = 0$ [7].

In a clean superconductor $[U(\mathbf{r},z) = 0]$, a flux line can be thought of as the world line of a 2D random walker, wandering in a plane with a removed disk of radius R_0. The winding number probability distribution for such a system is known to be Gaussian: $\mathcal{P}(n,L) \sim \exp(-x^2)$, with $x \sim n/\ln L$ [8]. Therefore,

$$\langle n^2(L)\rangle \sim \ln^2 \frac{L}{\lambda}. \quad (5)$$

Here we took into account the fact that the ultraviolet cutoff for the directed random walk description of flux lines is provided by the London penetration depth λ.

In the case of point disorder, a flux line represents the trajectory of a Markovian random walk with the mean-square displacement growing as $\langle r^2(L,T)\rangle \sim L^{2\nu}$, where $\nu \simeq 0.6$ in 2 + 1 dimensions [2]. Therefore, it is reasonable to argue that, at scales larger than the collective pinning (or Larkin) length L_c, where the disorder-induced wandering dominates [2], the properties of flux lines are statistically equivalent to those of Lévy flights [9] with index $\mu = 1/\nu$. This analogy allows one to calculate the vortex entanglement using the method of Ref. [10]. The mean-square winding number variation for a trajectory of length l after one elementary step of length $dl \simeq L_c$ is given by $d\sigma \equiv \langle(\delta n)^2\rangle = \int d\delta\, n(\delta n)^2 P(\delta n)$, where the distribution function is

$$P(x) = \int d\mathbf{r}\, d\mathbf{r}'\, \delta\left(x - \frac{1}{2\pi}\arccos\frac{\mathbf{r}\mathbf{r}'}{rr'}\right) \times f_\mu(\mathbf{r},l) f_\mu(\mathbf{r}' - \mathbf{r}, dl).$$

Here $f_\mu(\mathbf{r},l) = (1/4\pi^2)\int d\mathbf{k}\exp(i\mathbf{k}\mathbf{r} - |\mathbf{k}|^\mu l)$ is the Lévy distribution function. Calculating the integrals, we obtain $d\sigma(l) \sim dl/l$ (at $\mu < 2$). The angular variations at different steps are statistically independent, so that the total winding number variance after $L/L_c \gg 1$ steps is given by

$$\langle n^2(L)\rangle = \int_{L_c}^{L} d\sigma(l) \sim \ln\frac{L}{L_c}. \quad (6)$$

Some scaling and numerical arguments in favor of a logarithmic dependence of the vortex entanglement in a white-noise potential have been put forward in Ref. [11]. The result (6) is valid as long as the typical displacement of a single vortex ($\sim L^\nu$) is smaller than $n_v^{-1/2}$.

Now let $U(\mathbf{r}) = U_0 \sum_i \delta(\mathbf{r} - \mathbf{r}_i)$ be the potential of columnar defects parallel to the external field, whose positions $\mathbf{r}_i$ are distributed uniformly, according to the Poisson law with mean density ρ. This potential can be either attractive or repulsive. The former possibility is realized in HTSC where columnar defects can be artificially created by irradiating the material with high-energy ions [12]. Such defects are of the size of the order of 50–70 Å (which is comparable with the vortex core diameter) and tend to localize vortices in their vicinity. As we shall see shortly, the sign of $U(\mathbf{r})$ is not important for us, as long as the spectrum of the Hamiltonian (4) is bounded. For this reason, we assume that $U_0 > 0$, thus putting the lower spectrum boundary at zero energy.

The constrained partition function (3) can be represented in the following form:

$$Z_n(\mathbf{r},\mathbf{r};L) = \int dE\, e^{-EL} \int_0^{2\pi} \frac{d\phi}{2\pi}\, e^{i\phi n} N(E,\mathbf{r};\phi\,|\,U), \tag{7}$$

where $N(E,\mathbf{r};\phi\,|\,U)$ is the local density of eigenstates of the Hamiltonian (4) in a given configuration of disorder. At large L, the main contribution to the integral on the right-hand side of Eq. (7) comes from the energies near the lower spectrum boundary, where the eigenfunctions are localized [13]. In this limit, we obtain from (7)

$$\begin{aligned}\langle n^2(L)\rangle &= -\frac{\partial^2}{\partial\phi^2}\left\langle \ln\int dE\, e^{-EL} N(E,\phi|U)\right\rangle \Big|_{\phi=0} \\ &= -\frac{\partial^2}{\partial\phi^2} \ln\int dE\, e^{-EL} N(E,\phi)\Big|_{\phi=0}, \end{aligned} \tag{8}$$

where $N(E,\phi) = \int (d\mathbf{r}/\Omega)\langle N(E,\mathbf{r};\phi\,|\,U)\rangle$ is the average total density of states (DoS) (Ω is the system area in the plane perpendicular to the external field). To prove the last equality, one could use the replica trick $[\langle \ln x\rangle = \lim_{m\to 0}(\langle x^m\rangle - 1)/m]$ and notice that there are no level correlations in the localized phase, so that $\langle N(E_1,\phi\,|\,U)\cdots N(E_m,\phi\,|\,U)\rangle \to N(E_1,\phi)\cdots N(E_m,\phi)$.

The asymptotic behavior of the mean-square winding number at large L is thus determined by the asymptotics of the average density of states $N(E,\phi)$ at small E. This asymptotics ("Lifshitz tail") can be calculated by extending the well-known argumentation of Lifshitz [14] to the case of a nonzero magnetic flux. The basic idea is that the low-energy behavior of DoS is dominated by the contribution from large regions in real space which are free of defects. In the absence of solenoid, the low-lying eigenvalues for the wave functions localized inside such a region of area $A = \pi R^2$ coincide with the energy levels of a quantum particle in a two-dimensional potential well of radius R with infinitely high walls. In particular, the ground state energy is given by $E(R) = a^2T/2\epsilon_l R^2$ [where $a \simeq 2.405$ is the first root of the Bessel function $J_0(x)$] [15], so that $A(E) = \pi a^2 T/2\epsilon_l E$. On the other hand, the probability to find a clean region of area A is exponentially small: $p(A) \sim \exp(-\rho A)$, thus giving $N(E) \sim p[A(E)] \sim \exp(-\pi a^2\rho T/2\epsilon_l E)$ [14]. In the presence of a cylindrical obstacle of radius R_0 threaded by a solenoid, the flux dependence of the low-energy tail of DoS can be derived from that of the ground state energy in an annular potential well. The inner radius of the well is R_0, while the outer one is determined by the size of an optimal fluctuation in the concentration of defects: $R = (a^2T/2\epsilon_l E)^{1/2} \gg R_0$. At small ϕ, the first order perturbative correction to E reads

$$\delta E(\phi) = \frac{\int_{R_0}^R r\,dr\, \frac{T\phi^2}{8\pi^2\epsilon_l r^2}\, \psi_0^2(r)}{\int_{R_0}^R r\,dr\, \psi_0^2(r)}. \tag{9}$$

Here $\psi_0(r)$ is the ground state wave function in the absence of solenoid, satisfying the boundary conditions $\psi_0(R_0) = \psi_0(R) = 0$:

$$\psi_0(r) \sim J_0\left(\frac{ar}{R}\right) + \frac{\pi}{2}\left(\ln\frac{R}{R_0}\right)^{-1} Y_0\left(\frac{ar}{R}\right),$$

where $J_0(x)$ and $Y_0(x)$ are the Bessel functions. Calculating the integrals in Eq. (9), we obtain $\delta E(\phi) = b_0\phi^2 E\ln(E_0/E)$, where $b_0 = [12\pi^2a^2J_1^2(a)]^{-1}$, and $E_0 = a^2T/2\epsilon_l R_0^2$. To keep the ground state energy fixed, one has to compensate for this correction by increasing the area of a clean region: $\delta A(E,\phi) = (\pi a^2 T/2\epsilon_l E^2)\delta E(\phi)$, so that the DoS at a fixed energy decreases: $N(E,\phi) \sim \exp\{-\rho[A(E) + \delta A(E,\phi)]\}$. Finally, we obtain, with exponential accuracy,

$$N(E,\phi) = N(E,\phi = 0)\exp\left\{-b\,\frac{\rho T}{\epsilon_l E}\ln\frac{E_0}{E}\,\phi^2\right\}, \tag{10}$$

where $b = \pi a^2 b_0/2 \simeq 0.049$. The heuristic arguments leading to Eq. (10) can be confirmed by a rigorous analysis whereby the optimal fluctuations in the distribution of defects able to sustain the eigenstates with very low energy have been shown to correspond to the saddle-point solutions (instantons) in the field-theoretical formulation of the problem [16,17]. Calculating the integral over E in Eq. (8) by the steepest descent method, we finally obtain

$$\langle n^2(L)\rangle = c\left(\frac{\rho TL}{\epsilon_l}\right)^{1/2} \ln\frac{TL}{\rho\epsilon_l R_0^4}, \tag{11}$$

where $c = (b/a)\sqrt{2/\pi} \simeq 0.016$. The difference between the asymptotic winding number distributions of closed $[\mathbf{r}(0) = \mathbf{r}(L)]$ and open $[\mathbf{r}(0) \neq \mathbf{r}(L)]$ trajectories is determined by the coordinate dependence of the Green functions of the Hamiltonian (4), which affects only the preexponential factors and is therefore irrelevant at large L. To estimate the limits of applicability of Eq. (11), one has to compare the typical distance of wandering at length L with the mean distances $\rho^{-1/2}$ between the defects and $n_v^{-1/2}$ between the vortices. The mean-square displacement of a single vortex line due to thermal fluctuations can be derived using the analogy with a 2D random walk in the presence of static traps: $\langle r^2(L,T)\rangle \sim (TL/\rho\epsilon_l)^{1/2}$ [18,19]. As a result, we obtain

$$\frac{\epsilon_l\Phi_0}{B_\Phi} < TL < \frac{\epsilon_l\Phi_0}{B_\Phi}\left(\frac{B_\Phi}{B}\right)^2, \tag{12}$$

where $B_\Phi = \rho\Phi_0$ is the so-called matching field [2], which is typically between 1 and 5 T in experiment. The first inequality guarantees that one deals with a many-defect problem, while the second one allows us to neglect the intervortex interactions and can be most easily satisfied in the vicinity of H_{c1}. It follows from Eq. (12) that there exists a crossover length $L_{\rm cr} \sim \epsilon_l \Phi_0 / B_\Phi T$, which separates two different regimes of entanglement. At $L < L_{\rm cr}$, it is possible to neglect completely the influence of columnar disorder, and the typical winding number grows logarithmically with L; see Eq. (5). In contrast, at $L > L_{\rm cr}$ the dominant contribution comes from the DoS "tails," and the winding number grows much faster.

In conclusion, we found that quenched disorder strongly affects the topological entanglement of magnetic flux lines in multiply connected superconductors. While in the clean case the winding number grows logarithmically with the system size: $\langle n^2(L)\rangle_{\rm clean} \sim \ln^2 L$, a point disorder increases vortex wandering and therefore suppresses entanglement: $\langle n^2(L)\rangle_{\rm point} \sim \ln L$. In contrast, a columnar disorder tends to confine vortices inside the optimal fluctuations in the distribution of defects, thus decreasing transverse wandering and substantially increasing entanglement: $\langle n^2(L)\rangle_{\rm column} \sim L^{1/2} \ln L$.

The topological contribution to the effective pinning force can be estimated as follows. If a vortex is subject to an external Lorentz force, it can disentangle itself from an obstacle by a complex movement ("reptation") of the vortex line ends in the top and bottom planes of a sample. However, the characteristic times of disentanglement in this case grow faster than exponentially with the sample size [4], so that, in reality, the dominant contribution to the depinning rate might come from the "cutting" of vortex cores through the obstacle. The energy barrier for such a process is proportional to the winding number: $E_{\rm cut}(n) = E_1|n|$, where the energy cost of a single act of cutting is of the order of the creation energy of a closed vortex loop of radius R_0: $E_1 \simeq 2\pi R_0 (\Phi_0/4\pi\lambda)^2 \ln[\min(R_0, \lambda)/\xi]$. The critical current density j_c can then be estimated by balancing $E_{\rm cut}$ against the energy gain due to the action of Lorentz force $f_L = \Phi_0 j_c L/c$. Therefore,

$$\frac{j_c}{j_0} \sim \frac{\xi}{L} [\langle n^2(L)\rangle]^{1/2}, \tag{13}$$

where $j_0 \sim c\Phi_0/\lambda^2 \xi$ is the critical pair-breaking current density (ξ is the correlation length). For instance, in the presence of columnar defects [see Eq. (11)] the ratio (13) can be represented as $j_c/j_0 \sim (L_{\rm wind}/L)^{3/4} \ln L$, where $L_{\rm wind} = (\rho T \xi^4/\epsilon_l)^{1/3}$ is the characteristic length scale related to entanglement. Although j_c decreases with the sample size, the topological mechanism of pinning becomes increasingly effective as temperature grows and may dominate the conventional pinning (due to the localization of vortices in the effective potential wells created by defects) in sufficiently thin samples at high temperatures.

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