Helicity Redistribution during Relaxation of Astrophysical Plasmas

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We present the first 3D numerical MHD simulations that show that Taylor's relaxation conjecture is not satisfied in some MHD evolution of magnetic configurations encountered in solar physics. We show that magnetic helicity can be slowly injected through the boundary into a magnetic configuration which then evolves into a MHD disruption, with the formation in finite time of a current sheet through which reconnection occurs, leading to a release of magnetic energy. While helicity is well conserved during the process, it is shown that the relaxed state is far from the constant-α linear force-free field that would be predicted by Taylor's conjecture.

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In various situations of laboratory and space plasmas the magnetic Reynolds number is very large. Magnetic helicity can be considered as a conserved quantity [1]. It was conjectured in [2] that after a major disruption phase the system should relax towards a constant-α force-free field, with a predicted current profile. This conjecture turned out to be quite impressively satisfied in the case of this reversed field pinch ZETA machine [3].

The situation was found to be different in the tokamaks, for which the prediction of the theory did not match the measured current profiles [4]. Although there are major differences between the laboratory and solar cases, the latter deals in general with either bounded or unbounded domains for which the magnetic field threads the boundary (i.e., the boundary is not a flux surface); the conjecture has been extensively used in many situations dealing with astrophysical plasma systems and, in particular, in solar coronal physics such as solar flare and coronal heating theory [5].

For such astrophysical configurations whose footpoints are rigidly anchored in the boundary ($B_n|_{\partial\Omega} \neq 0$), one needs to define a new helicity that is gauge invariant [6]. There is no unique definition, and one can take $\delta H = \int_\Omega (\mathbf{A} - \mathbf{A}_0) \cdot (\mathbf{B} + \mathbf{B}_0) d^3\mathbf{r} + \int_{\partial\Omega} \kappa(\mathbf{B} + \mathbf{B}_0) \cdot d\mathbf{s}$ [6,7], where $\mathbf{A}$ is a potential vector associated with $\mathbf{B}$, $\mathbf{B}_0$ is the unique magnetic field corresponding to the same boundary value $B_n|_{\partial\Omega}$, and $\mathbf{A}_0$ is an associated potential vector. κ may be explicitly given in terms of $\mathbf{B}$ and $\mathbf{B}_0$. This second integral vanishes when the domain is the upper half space. A variational problem can also be defined in which one minimizes the magnetic energy for a fixed value for this relative helicity, and the Woltjer theorem [1] can be generalized. When the minimizer corresponds to the solution of the Euler-Lagrange equation, it is the so-called constant-α force-free (or Beltrami) field solution of $\nabla \times \mathbf{B} = \alpha\mathbf{B}$ in Ω with α some *a priori* unknown constant appearing as the Lagrange multiplier [8]. General existence and uniqueness theorems as well as particular classes of solutions for this equation in bounded and unbounded domains and various geometries have been extensively investigated (Ref. [9], and references therein). It is not clear whether the Taylor state is the actual relaxed state for such configurations whose footpoints are anchored in the boundary.

In this Letter we address the following issue (relevant to the physics of the solar corona and therefore of astrophysical interest). One assumes an almost infinitely conducting low beta plasma and a situation different from the laboratory ($B_n|_{\partial\Omega} \neq 0$) in a bounded (or unbounded) domain, and a magnetic configuration similar to arcadelike flux tubes threading the "solar surface." If helicity is injected through the boundary and then a major disruption occurs while magnetic helicity conservation is fulfilled to a relatively good extent, what will be the nature of the relaxed state? According to Taylor's conjecture, if it would apply, this final state should be a constant-α linear force-free field.

Addressing this issue is difficult because of the need to perform three-dimensional MHD simulations in which a magnetic configuration embedded in a highly conductive low-β plasma would slowly be driven up to an instability phase and then to a relaxation phase. The formation of a singularity in finite time, initiating reconnection, is also an important related issue [10].

This Letter presents the first 3D numerical MHD simulations which lead a magnetic configuration to develop near singularities (current sheets) and initiate reconnection and relaxation to an equilibrium state. For previous initial conditions new simulations are performed [11]. Here it is shown that the magnetic helicity of the system increases linearly during the slow evolution phase driven by boundary motions, and that helicity is well conserved during the relaxation phase associated with reconnection when the boundary motions are switched off. We show that the relaxed state is far from the constant-α linear force-free state that would be predicted by Taylor's conjecture and,

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in particular, that magnetic helicity is redistributed in the current sheets during the relaxation process.

Our code is based on a three-dimensional numerical scheme whose time advance is semi-implicit or implicit, allowing the use of large time steps and is nonperiodic in the three dimensions of the Cartesian space. This code is an extension of the axis-symmetric version developed recently [12]. The scheme uses an unstaggered mesh that allows zero divergence of the magnetic field up to round-off accuracy. We solve the full set of magnetohydrodynamic equations for the velocity $\mathbf{v}$, magnetic field $\mathbf{B}$, density ρ, and pressure p. We use small values for the dissipation coefficients: $\nu = 10^{-2}$–10^{-3} for the kinematic viscosity and $\eta = 10^{-5}$ for the plasma resistivity. Since we are dealing with astrophysical situations such as in the solar corona, β (the ratio of the magnetic to the gas pressure) is small (of the order of 10^{-3} or even smaller); typical simulations were done with these values or with $\beta = 0$, and did not show differences. The computational domain is the finite box $\{0 < x < L_x, 0 < y < L_y, 0 < z < L_z\}$, whose size is large compared to the characteristic spatial scale of the system (which is our reference length). We choose $L_x = L_y = L_z = 40$–60. The MHD equations are then discretized on a nonuniform mesh ($111 \times 101 \times 70$ nodes). Note that there exists a residual numerical dissipation which is ($\eta = 10^{-6}$–10^{-5}) smaller or of the order of the value taken for the actual resistivity ($\eta = 10^{-5}$), but never higher.

As an initial condition at $t = 0$ we take $\mathbf{B}$ to be the bipolar potential field (i.e., current free) such that $B_z(y, 0)$ is represented by two elliptic 2D Gaussian functions. This configuration represents an arcade above the half plane such that the normal component $B_n|_{\partial\Omega} \neq 0$, unlike for typical magnetic fusion configurations such as in the tokamak as shown in Fig. 1a.

For $t \geq 0$, one injects helicity through the boundary by imposing a 2D velocity field $\mathbf{v}_b$ at the bottom of the box (and no velocity field on the other boundaries). This boundary velocity field is chosen to correspond to two parallel vortices rotating in the same direction and located on each side of the line where $B_z(y, 0) = 0$. $\max|\mathbf{v}_b| = 10^{-2}$ (i.e., small compared to the Alfvén speed $v_A = 1$ in our units). We chose the support of the vortices to be small enough so that these are imposed on a small fraction of the feet of magnetic configuration that defines the flux rope shown in Fig. 1a. The evolution is therefore that of a confined magnetic flux rope that will be twisted by this boundary velocity field while the overlaying confining arcade will remain untwisted.

During a first phase which lasts up to about $t = 155$–170 the configuration evolves through a sequence of force-free equilibria (quasistatic evolution at a speed slower than the driving boundary velocity). The magnetic configuration builds up a considerable amount of self-helicity [corresponding to a twist of about $(2.3$–$2.7)\pi$] and magnetic energy in the inner twisted flux tube, while the overlaying confining arcade remains almost current

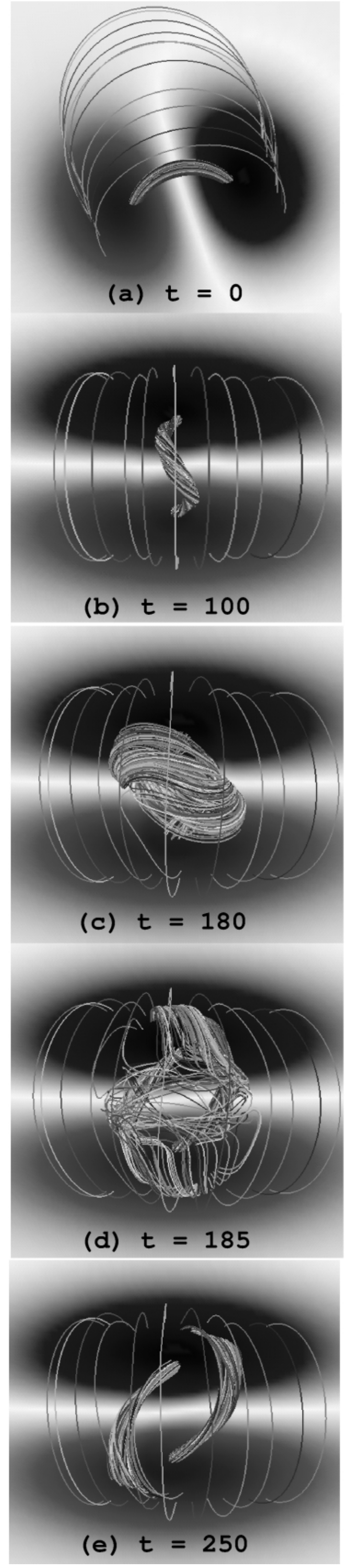

FIG. 1. MHD evolution of the magnetic structure corresponding to helicity injection by twisting motions during a first quasistatic phase [(a),(b)], followed by a dynamic phase when the boundary motions are switched off [(c)–(e)]. The simulation time (in units of Alfvén time) is indicated at the bottom of each frame.

free. By switching off the boundary vortex motions several times during this first phase and performing a viscous relaxation, the configuration reaches a state of numerical equilibria.

During a second phase starting after ($t \sim 170$) the system evolves through a dynamic phase during which the configuration experiences a major disruption, as shown in Figs. 1c–1e. Switching off $\mathbf{v}_b$ we found no neighboring accessible equilibria via viscous relaxation, unlike in the first phase. In a few tens of Alfvén times, the field lines of the flux tube reconnect with those of the overlaying structure.

Reconnection actually occurs through an evolving current sheet located on the interface between the twisted flux tube and the overlaying structure. This current sheet [13] is the generalization to 3D of the one obtained in cylindrical geometry due to the kink instability [14]. This current sheet coincides with where the magnitude of the velocity is higher during the reconnection phase (slightly smaller or of the order of the Alfvén speed). Despite the limited resolution of the simulation, this result shows an increase of peak current consistent with the development of a singularity in finite time in MHD equations [10]. Figure 2 shows the evolution in time of sup$|\alpha|$ in a subdomain containing the evolving current sheet, where $\alpha = \frac{\mathbf{J}\cdot\mathbf{B}}{|\mathbf{B}|^2}$ with $\mathbf{J}$ representing the electric current. It is worth noticing that sup$|\alpha|$ is still increasing during the reconnection phase. This is explained by the fact that the driving process is an ideal instability (that has not yet saturated to a singular equilibrium) and therefore, as with the ideal kink instability in toroidal geometry, it is still building up electric currents even in the presence of resistivity.

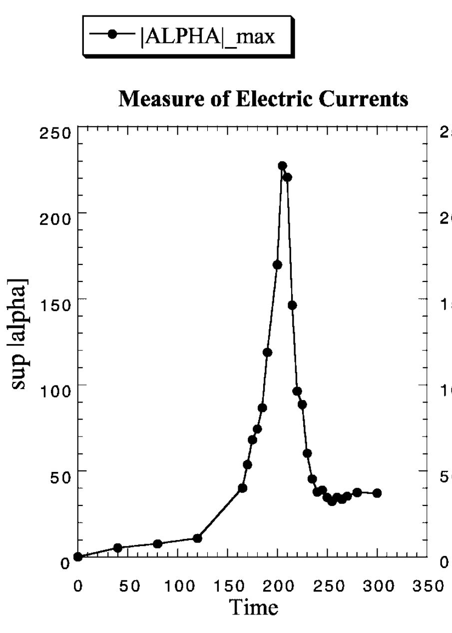

FIG. 2. Time evolution of the quantity $|\alpha|_{\max}$ showing that the system first evolves towards the formation of a current singularity up to the limit of our finite spatial resolution.

At the end of the relaxation phase (Fig. 1e) the configuration reached an equilibrium state in which the initially single twisted flux tube split into two almost untwisted flux tubes. This result is again a generalization to 3D of the results of [14].

Magnetic energy is injected as mechanical energy on the boundary and stored during the first phase of helicity injection, as shown on Fig. 3 (in units of the initial potential magnetic field energy). Then during the second disruption phase a non-negligible part of the free magnetic energy is released.

A property of our numerical code (due to the mesh staggering) is that the relative magnetic helicity defined above is conserved up to round-off accuracy. Figure 3 shows the evolution of the relative magnetic helicity. During the first phase of helicity injection, magnetic helicity increases linearly as would be expected from any injection from a stationary velocity field: $dH/dt = 2\int_{\partial\Omega}(\mathbf{A}_0\cdot\mathbf{B})(\mathbf{v}\cdot\mathbf{n}) - (\mathbf{v}\cdot\mathbf{A}_0)(\mathbf{B}\cdot\mathbf{n})\,d\mathbf{s} - 2\int_{\Omega}(\mathbf{J}\cdot\mathbf{B})/\sigma\,d^3\mathbf{r}$, where σ is the conductivity of the medium and $\mathbf{n}$ is the normal unit vector directed onwards from the domain. The only source of injection of helicity is tangential fluid boundary motions, and, since electric conductivity is high enough, the first term becomes the only remaining one. Therefore Taylor's conjecture could, in principle, be applied and would predict a constant-α linear force-free field.

However, it is straightforward to check that the relaxed final state is still far from a linear force-free state, as shown

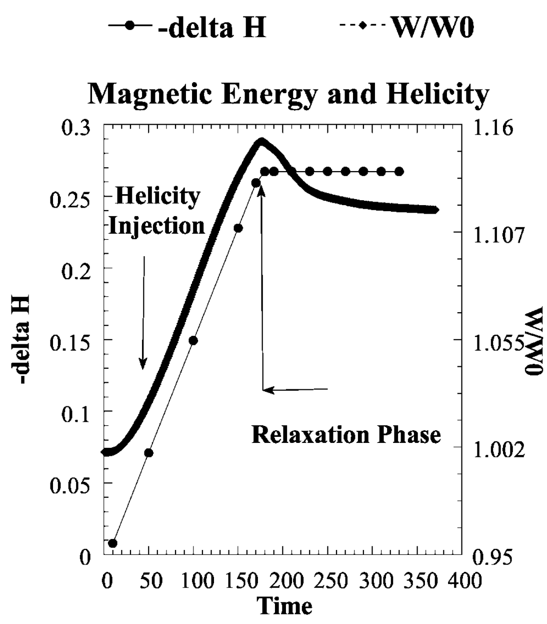

FIG. 3. Compared evolution of relative magnetic energy W/W_0 of the field (in units of the potential field energy) and relative magnetic helicity δH. Magnetic helicity is injected linearly in the driving phase and stored as self-helicity and then conserved during the relaxation phase while a non-negligible amount of magnetic energy is released during the relaxation phase.

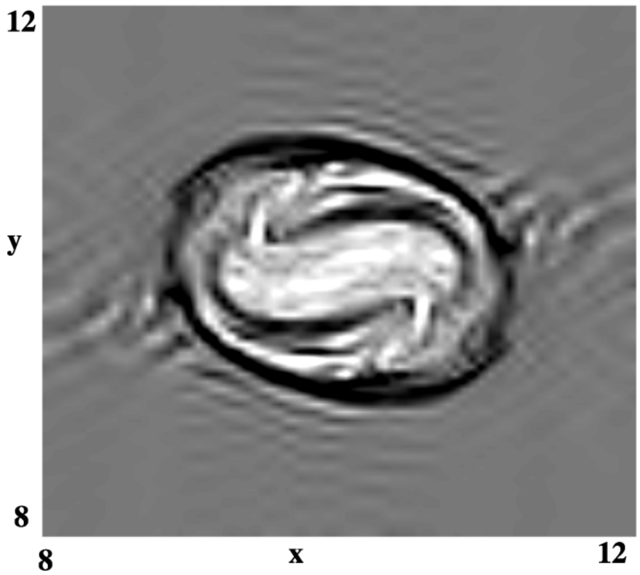

FIG. 4. Distribution of α in the relaxed state, showing that the system does not relax towards a constant-α force-free field as predicted by Taylor's conjecture.

in Fig. 4, since nonlinearities remain in the configuration. Magnetic helicity which was stored in the form of self-helicity has been redistributed: it has been ejected from the twisted flux tube (which is now broken into two untwisted flux tubes), towards the singularities (current sheets), and α almost vanishes at the location where the twist built up.

These results seem to put a limit on the conditions of applicability of Taylor's conjecture. In the absence of fully developed turbulence and in the presence of a current singularity, we conjecture that after a single coherent disruptive event *the system should relax towards a state in which helicity would be redistributed towards the boundary of the domain*, so that the relaxed state is not a constant-α force-free field.

From these results it seems that other constraints should be found and imposed to determine a variational problem that prevents the system from relaxing towards a constant-α force-free state. New directions should be investigated: (i) imposing part of magnetic helicity (corresponding to the flux tubes involved in the process) and not total magnetic helicity, (ii) defining a generalized action associated with the variational problem that incorporates the new boundary anchoring constraint that may prevent large scale homogenization up to the constant-α Taylor relaxed state, and/or (iii) looking for new higher order invariants of MHD equations that have the same properties of magnetic helicity, mainly to inverse cascade towards large scales.

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