

Tiling the Plane without Supersymmetry

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We present a way of tiling the plane with a regular hexagonal network of defects. The network is stable and follows in consequence of the three-junctions that appear in a model of two real scalar fields that presents Z_3 symmetry. The Z_3 symmetry is effective in both the vacuum and defect sectors, and no supersymmetry is required to build the network.

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Domain walls appear in diverse branches of physics, involving energy scales as different as the ones, for instance, in magnetic materials [1] and in cosmology [2]. They live in three spatial dimensions as bidimensional objects that arise in systems with at least two isolated degenerate minima. In field theory they appear in the (3,1) dimensional space-time, and this may happen in supersymmetric theories, although supersymmetry plays no fundamental role in the presence of domain walls.

Very recently, in a paper by Gibbons and Townsend [3], and also in Refs. [4,5], one investigates the presence of domain walls and their possible intersections in a Wess-Zumino model, with a polynomial superpotential. In the supersymmetric theory, one can classify the classical solutions as BPS and non-BPS states, according to the work of Bogomol'nyi, and of Prasad and Sommerfield [6]. The BPS states are stable, and are expected to play some role in investigating duality in supersymmetric models. We recall that no BPS state can be annihilated under continuum variation of the parameters that define the supersymmetric theory.

In Ref. [7] one investigates models of coupled real scalar fields in bidimensional space-time. These investigations provide a concrete way of finding BPS states and suggest other studies, in particular on the subject of defects inside defects; see Ref. [8]. Most of the models investigated in [7,8] can be seen as real bosonic portions of supersymmetric theories. In supersymmetric models the presence of discrete symmetry may produce BPS and non-BPS defects. The BPS states lie in shorter multiplets, and preserve the supersymmetry only partially [9,10]. There are BPS states that preserve 1/2 of the supersymmetry, but the possibility of BPS states preserving 1/4 supersymmetry is subtler, and is shown to appear as junctions [11,12] of domain walls in the recent papers [3–5].

In the present work we start dealing with the bosonic portions of supersymmetric theories. We do this guided by the discrete Z_3 symmetry, with the aim of describing the presence of three-junctions and the network of defects that it can generate. We first point out that supersymmetry introduces restrictions that may lead to instability of the junction, or at least of the network that it could generate. We then examine another model, and show that all the

difficulties found in the supersymmetric context are circumvented by just giving up supersymmetry.

The subject of this work may be of interest to several different branches of physics, in particular in applications concerning the entrapment of networks of defects inside domain walls. This possibility can be implemented with three scalar fields, in models engendering the $Z_2 \times Z_3$ symmetry, following the lines of Refs. [8,13]. Other applications may include strong interactions, if we recall that the Z_3 group is the center of the $SU(3)$ group; see, for instance, Refs. [14,15]. Also, there are applications to systems of condensed matter, in particular on issues concerning pattern formation [16], as for instance in the case of the thermal convection studied in Ref. [17].

We start describing the two real scalar fields ϕ and χ in bidimensional space-time with the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi + \frac{1}{2} \partial_\alpha \chi \partial^\alpha \chi - V. \quad (1)$$

Here $V = V(\phi, \chi)$ is the potential. In the supersymmetric case it has the general form

$$V(\phi, \chi) = \frac{1}{2} W_\phi^2 + \frac{1}{2} W_\chi^2, \quad (2)$$

where $W = W(\phi, \chi)$ is the superpotential.

The superpotential allows introducing several properties, as shown in Refs. [7,8]. For instance, for static fields the equations of motion are solved by first-order differential equations $d\phi/dx = W_\phi$ and $d\chi/dx = W_\chi$. The energy of solutions of the first-order equations are given by $E_B^{ij} = |\Delta W_{ij}|$, with $\Delta W_{ij} = W_i - W_j$ and $W_i = W(\phi_i, \chi_i)$, where (ϕ_i, χ_i) represents the i th vacuum state of the model. This is the Bogomol'nyi bound, and the corresponding solutions are BPS solutions. The BPS solutions are linearly or classically stable.

We guide ourselves toward the topological solutions by introducing the topological current

$$J^\alpha = \varepsilon^{\alpha\beta} \partial_\beta \begin{pmatrix} \phi \\ \chi \end{pmatrix}. \quad (3)$$

It obeys $\partial_\alpha J^\alpha = 0$, and it is also a vector in the (ϕ, χ) plane. For static configurations we have $J_\alpha^t J^\alpha = \rho^t \rho$, where $\rho = \rho(\phi, \chi)$ is the charge density. This charge density allows writing $\rho^t \rho$ as twice the kinetic energy

density of the topological solution, and this can be used to infer stability of junctions; see below.

Let us now consider a specific model, defined by the superpotential [13]

$$W(\phi, \chi) = \lambda\phi^3 - 3\lambda\phi + \lambda\phi\left(\chi + \sqrt{\frac{1}{3}}\right)^2 + \lambda\sqrt{\frac{4}{27}}\left(\chi + \sqrt{\frac{1}{3}}\right)^3 - \lambda\left(\chi + \sqrt{\frac{1}{3}}\right)^2. \quad (4)$$

The first-order differential equations are given by

$$\frac{d\phi}{dx} = 3\lambda(\phi^2 - 1) + \lambda\left(\chi + \sqrt{\frac{1}{3}}\right)^2, \quad (5)$$

$$\frac{d\chi}{dx} = 2\lambda\left(\chi + \sqrt{\frac{1}{3}}\right)\left[\phi + \sqrt{\frac{1}{3}}\left(\chi + \sqrt{\frac{1}{3}}\right) - 1\right]. \quad (6)$$

The model is described by the fourth-order polynomial potential

$$V_4(\phi, \chi) = \frac{9}{2}\lambda^2(\phi^2 - 1)^2 + \frac{1}{2}\lambda^2\left(\chi + \sqrt{\frac{1}{3}}\right)^4 + 3\lambda^2(\phi^2 - 1)\left(\chi + \sqrt{\frac{1}{3}}\right)^2 + 2\lambda^2\left(\chi + \sqrt{\frac{1}{3}}\right)^2\left[\phi + \sqrt{\frac{1}{3}}\chi - \frac{2}{3}\right]^2. \quad (7)$$

Thus, it behaves standardly in one, two, and three spatial dimensions.

The above potential presents the three vacuum states $(0, \sqrt{4/3})$, $(-1, -\sqrt{1/3})$, and $(1, -\sqrt{1/3})$. These minima form an equilateral triangle invariant under the Z_3 rotations, with side $l = 2$. The values of the superpotential at the minima are $W_1 = -\lambda$, $W_2 = 2\lambda$, and $W_3 = -2\lambda$. The energies of the BPS states are $|\lambda|$, $3|\lambda|$, and $4|\lambda|$. All the three sectors are BPS sectors, but they do not present the Z_3 symmetry that connects the vacuum states. The highest energy is associated to the sector connecting the second and third vacuum states. This is the only sector where we can find explicit solutions. They are given by $\phi(x) = -\tanh(3\lambda x)$ and $\chi(x) = -\sqrt{1/3}$. This is a BPS state, representing an orbit in the (ϕ, χ) plane. The orbit is a straight-line segment that connects the corresponding vacuum states. The orbits connecting the other vacua cannot be straight-line segments. They cannot be obtained by rotating the (ϕ, χ) plane according to the Z_3 symmetry, and so the defect sectors do not present the Z_3 symmetry that connects the vacuum states. This fact also appears when one identifies the tensions of the BPS defects. They are given by $t_1 = |\lambda|$, $t_2 = 3|\lambda|$, and $t_3 = 4|\lambda|$. They are different and do not obey the Z_3 symmetry. They are

such that $t_3 = t_1 + t_2$, and do not strictly obey the triangle inequality one needs to ensure stability [12] of the three-junction that appears in this model.

To circumvent instability of the three-junction we now follow Refs. [3,4]. We make contact with these works after considering superpotentials that satisfy $W_{\phi\phi} + W_{\chi\chi} = 0$. In this case, for harmonic superpotentials one adds to the two first-order equations $d\phi/dx = W_\phi$ and $d\chi/dx = W_\chi$ the two new [13] first-order equations: $d\phi/dx = -W_\chi$ and $d\chi/dx = W_\phi$. Solutions to these equations also minimize the energy and solve the equations of motion. This allows introducing $\tilde{W}(\phi, \chi)$ such that $\tilde{W}_\phi = -W_\chi$ and $\tilde{W}_\chi = W_\phi$. We use W and \tilde{W} to introduce the complex superpotential, $\mathcal{W} = W + i\tilde{W}$. We write the complex superpotential in terms of the complex field $\phi + i\chi$, and this is the way one gets from the investigations of Refs. [7,8] to the recent possibility [3–5] of describing three-junctions preserving 1/4 supersymmetry. However, junctions require the presence of at least three minima, and this is achieved only when the superpotential is of at least the fourth-order power in the complex field. This means that the model behaves standardly only in one and two spatial dimensions. In this case one can show explicitly [3,4] that the three-junction is stable and breaks 1/4 supersymmetry, although supersymmetry itself does not allow the presence of a stable network of defects [3–5]. Owing to the fact that each adjacent junction in the network has opposite winding number, any adjacent vacua should be connected with defect solutions also having opposite winding numbers along the same orbit [5]. Since we have to use different conjugate Bogomol'nyi equations to take into account these winding numbers, the network clearly cannot be BPS and then can decay.

We then give up supersymmetry, turning our attention to polynomial potentials that engender the Z_3 symmetry, and that support stable three-junctions that generate a regular hexagonal network of defects. Interestingly, we have found a fourth-order polynomial potential that does the job. It is given by

$$V(\phi, \chi) = \lambda^2\phi^2\left(\phi^2 - \frac{9}{4}\right) + \lambda^2\chi^2\left(\chi^2 - \frac{9}{4}\right) + 2\lambda^2\phi^2\chi^2 - \lambda^2\phi(\phi^2 - 3\chi^2) + \frac{27}{8}\lambda^2. \quad (8)$$

This potential was introduced in Ref. [18]. The equations of motion for static configurations are

$$\frac{d^2\phi}{dx^2} = \lambda^2\phi\left(4\phi^2 + 4\chi^2 - 3\phi - \frac{9}{2}\right) + 3\lambda^2\chi^2, \quad (9)$$

$$\frac{d^2\chi}{dx^2} = \lambda^2\chi\left(4\phi^2 + 4\chi^2 + 6\phi - \frac{9}{2}\right). \quad (10)$$

The potential has three degenerate minima, at the points $v_1 = (3/2)(1, 0)$ and $v_{2,3} = (3/4)(-1, \pm\sqrt{3})$. These

minima form an equilateral triangle, invariant under the Z_3 symmetry. The distance between the minima is $(3/2)\sqrt{3}$.

We can obtain the topological solutions explicitly. The easiest way to do this follows by first examining the sector that connects the vacua v_2 and v_3 . This is so because in this case we set $\phi = -3/4$, searching for a straight-line segment in the (ϕ, χ) plane. This is compatible with Eq. (9), and reduces the other equation (10) to the form

$$\frac{d^2\chi}{dx^2} = \lambda^2 \left(4\chi^3 - \frac{27}{4}\chi \right). \quad (11)$$

This implies that the orbit connecting the vacua v_2 and v_3 is a straight line. It is such that along the orbit the χ field feels the potential $\lambda^2[\chi^2 - (27/16)]^2$. This shows that the model reduces to a model of a single field, and the solution satisfies the first-order equation

$$\frac{d\chi}{dx} = \sqrt{2}\lambda \left(\chi^2 - \frac{27}{16} \right). \quad (12)$$

The solution is

$$\chi(x) = -\frac{3}{4}\sqrt{3} \tanh\left(\sqrt{\frac{27}{8}}\lambda x\right). \quad (13)$$

The other solutions can be obtained by rotations obeying the Z_3 symmetry of the model.

The full set of solutions of the equations of motion are collected below. In the sector connecting the minima v_2 and v_3 they are

$$\phi_{(2,3)}^{(\pm)} = -\frac{3}{4}, \quad (14)$$

$$\chi_{(2,3)}^{(\pm)} = \pm\frac{3}{4}\sqrt{3} \tanh\left(\sqrt{\frac{27}{8}}\lambda x\right). \quad (15)$$

In the sector connecting the minima v_1 and v_2 they are

$$\phi_{(1,2)}^{(\pm)} = \frac{3}{8} \pm \frac{9}{8} \tanh\left(\sqrt{\frac{27}{8}}\lambda x\right), \quad (16)$$

$$\chi_{(1,2)}^{(\pm)} = \frac{3}{8}\sqrt{3} \mp \frac{3}{8}\sqrt{3} \tanh\left(\sqrt{\frac{27}{8}}\lambda x\right). \quad (17)$$

In the sector connecting the minima v_1 and v_3 they are

$$\phi_{(1,3)}^{(\pm)} = \frac{3}{8} \mp \frac{9}{8} \tanh\left(\sqrt{\frac{27}{8}}\lambda x\right), \quad (18)$$

$$\chi_{(1,3)}^{(\pm)} = -\frac{3}{8}\sqrt{3} \mp \frac{3}{8}\sqrt{3} \tanh\left(\sqrt{\frac{27}{8}}\lambda x\right). \quad (19)$$

The label (\pm) is used to identify kink and antikink. All the solutions have the same energy, $(9/4)\sqrt{27/8}|\lambda|$.

We examine how the bosonic fields behave in the background of the classical solutions. We do this by considering fluctuations around the static solutions $\phi(x)$ and $\chi(x)$.

We use the equations of motion to see that the fluctuations depend on the potential

$$\mathbf{U}(x) = \begin{pmatrix} V_{\phi\phi} & V_{\phi\chi} \\ V_{\chi\phi} & V_{\chi\chi} \end{pmatrix}. \quad (20)$$

Evidently, after obtaining the derivatives we substitute the fields by their classical static values $\phi(x)$ and $\chi(x)$. The model under consideration is defined by the potential (8). In this case we use (14) and (15) to obtain two decoupled equations for the fluctuations. The potentials of the corresponding Schrödinger-like equations are

$$U_{11}(x) = \frac{27}{8}\lambda^2 \left[4 - 2\text{sech}^2\left(\sqrt{\frac{27}{8}}\lambda x\right) \right], \quad (21)$$

$$U_{22}(x) = \frac{27}{8}\lambda^2 \left[4 - 6\text{sech}^2\left(\sqrt{\frac{27}{8}}\lambda x\right) \right]. \quad (22)$$

The eigenvalues can be obtained explicitly: in the χ direction we get $w_0^\chi = 0$ and $w_1^\chi = (9/2)\sqrt{\lambda^2/2}$, and in the ϕ direction we have $w_0^\phi = (9/2)\sqrt{\lambda^2/2}$. This shows that the pair (14) and (15) is stable, and by symmetry all the three topological solutions are stable solutions.

The classical solutions present the nice property of having energy evenly distributed in their kinetic (k) and potential (p) portions. In terms of energy density they are

$$k(x) = p(x) = \frac{1}{4} \left(\frac{27}{8} \right)^2 \lambda^2 \text{sech}^4\left(\sqrt{\frac{27}{8}}\lambda x\right). \quad (23)$$

To understand this feature we recall the calculation done explicitly in the sector with $\phi = -3/4$, constant. There the model is shown to reduce to a model of a single field, a model that supports BPS solutions. Within this context, the above solutions are very much like the non-BPS solutions that appear in supersymmetric systems [13]. We use this property and the topological current (3) to obtain $\rho^i \rho = \varepsilon$, where $\varepsilon(x) = k(x) + p(x)$ is the (total) energy density of the solution. We use this result and the notation ij to identify the sector connecting the vacua (ϕ_i, χ_i) and (ϕ_j, χ_j) , to show that for any two different sectors ij and jk , $i, j, k = 1, 2, 3$, we get

$$(\rho_{ij} + \rho_{jk})^i (\rho_{ij} + \rho_{jk}) < \rho_{ij}^i \rho_{ij} + \rho_{jk}^i \rho_{jk}. \quad (24)$$

This condition shows that the three-junction is a process of fusion of defects that occurs exothermically, providing stability of junctions in the present model. This result is more general than the one in Ref. [12], which appears within the context of supersymmetry. Evidently, our result also works for BPS and non-BPS solutions that appear in supersymmetric models, with the property of having energy evenly distributed in their kinetic and potential portions [13].

We notice that the orbits corresponding to the stable defect solutions form an equilateral triangle in the (ϕ, χ) plane. This is so because the solutions are straight-line segments joining the three vacuum states in configuration

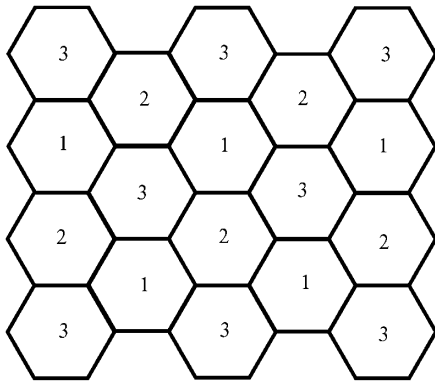


FIG. 1. A regular hexagonal network of defects, formed by three-junctions surrounded by domains representing the vacua $v_1 = 1$, $v_2 = 2$, and $v_3 = 3$.

space. They are degenerate in energy, and this allows associating to each defect the same tension,

$$t = \frac{9}{4} \sqrt{\frac{27}{8}} |\lambda|. \quad (25)$$

This makes $t_{ij} < t_{jk} + t_{ki}$, $i, j, k = 1, 2, 3$, and now the inequality is strictly valid in this case, stabilizing the three-junction that appears in this model when one enlarges the space-time to three spatial dimensions.

We consider the possibility of junctions in the plane, which may give rise to a planar network of defects. We work in the $(2, 1)$ space-time, in the plane (x, y) . We identify the plane (x, y) with the space of configurations, the plane (ϕ, χ) . We illustrate this situation by considering, for instance, the solutions we have already obtained. They are collected in Eqs. (14)–(19) in $(1, 1)$ dimensions. In the planar case they change to

$$\phi_{(2,3)}^{(\pm)} = -\frac{3}{4}, \quad (26)$$

$$\chi_{(2,3)}^{(\pm)} = \pm \frac{3}{4} \sqrt{3} \tanh\left(\sqrt{\frac{27}{8}} \lambda y\right), \quad (27)$$

and

$$\phi_{(1,2)}^{(\pm)} = \frac{3}{8} \mp \frac{9}{8} \tanh\left[\frac{1}{2} \sqrt{\frac{27}{8}} \lambda (y + \sqrt{3}x)\right], \quad (28)$$

$$\chi_{(1,2)}^{(\pm)} = \frac{3}{8} \sqrt{3} \pm \frac{3}{8} \sqrt{3} \tanh\left[\frac{1}{2} \sqrt{\frac{27}{8}} \lambda (y + \sqrt{3}x)\right], \quad (29)$$

and

$$\phi_{(1,3)}^{(\pm)} = \frac{3}{8} \pm \frac{9}{8} \tanh\left[\frac{1}{2} \sqrt{\frac{27}{8}} \lambda (y - \sqrt{3}x)\right], \quad (30)$$

$$\chi_{(1,3)}^{(\pm)} = -\frac{3}{8} \sqrt{3} \pm \frac{3}{8} \sqrt{3} \tanh\left[\frac{1}{2} \sqrt{\frac{27}{8}} \lambda (y - \sqrt{3}x)\right]. \quad (31)$$

These planar defects are domain walls, and can be used to represent the three-junction in the limit of thin walls.

The three-junction that appears in this Z_3 -symmetric model allows building a network of defects, precisely in the form of a regular hexagonal network, as depicted in Fig. 1 in the thin wall approximation. In this network the tension associated to the defect is the typical value of the energy in this tiling of the plane with a regular hexagonal network, which seems to be the most efficient way of tiling the plane. As we have shown, our model behaves standardly in $(3, 1)$ dimensions. It supports stable three-junctions that generate a stable regular hexagonal network of defects.

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