Nonlinear Supergravity on a Brane without Compactification

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We show that smooth domain wall spacetimes supported by a scalar field separating two anti–de Sitter–like regions admit a single graviton bound state. Our analysis yields a fully nonlinear supergravity treatment of the Randall-Sundrum model. Our solutions describe a pp-wave propagating in the domain wall background spacetime. If the latter is a Bogomol'nyi-Prasad-Sommerfeld state, our solutions retain some supersymmetry. Nevertheless, the Kaluza-Klein modes generate "pp curvature" singularities in the bulk located where the horizon of the anti–de Sitter region would ordinarily be.

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1. Introduction.—It has long been thought that any attempt to model the Universe as a single brane embedded in a higher-dimensional bulk spacetime must inevitably fail because the gravitational forces experienced by matter on the brane, being mediated by gravitons traveling in the bulk, are those appropriate to the higher-dimensional spacetime rather than the lower-dimensional brane. Recently, however, Randall and Sundrum [1,2] have argued that there are circumstances under which this need not be so. Their model involves a thin "distributional" static flat domain wall, or three brane, separating two regions of five-dimensional anti–de Sitter (AdS) spacetime. They solve for the linearized graviton perturbations and find a square integrable bound state representing a gravitational wave confined to the domain wall. They also found the linearized bulk or "Kaluza-Klein" graviton modes. They argue that the latter decouple from the brane and make negligible contribution to the force between two sources in the brane, so that this force is due primarily to the bound state. In this way we get an inverse square law attraction rather than the inverse cube law one might naively have anticipated (see [3] for a related discussion).

This result is rather striking and raises various questions. For example, one would like to know how general the effect is. It is just an effect of the linearized perturbations or does it persist when nonlinearities are taken into account? One would expect to get only one massless spin two bound state if the effective theory on the brane is to be general relativity. In their derivation a crucial role is played by a delta function in the linearized graviton equation of motion. This is responsible for the unique bound state. It also seems that the effect will only work for domain walls and not for other branes. However, the full dynamics of the domain wall is not treated in detail in the Randall-Sundrum model. In fact, gravitating domain walls have a drastic effect on the curvature of the ambient spacetime and it is not obvious that a simple model involving a single collective coordinate representing the transverse displacement of the domain wall is valid.

For these reasons it seems desirable to have a simple nonsingular model which is exactly solvable. It is the purpose of this Letter to provide that.

2. Thick domain walls in AdS domains.—We first seek a static domain wall solution of the d-dimensional Einstein equations

$$R_{mn} - \tfrac{1}{2}Rg_{mn} = \partial_m\Phi\partial_n\Phi - g_{mn}[\tfrac{1}{2}\partial_a\Phi\partial_b\Phi g^{ab} + V(\Phi)], \quad (2.1)$$

where $a, b = 0, 1, 2, \ldots, d-1$. The right-hand side of (2.1) is the energy momentum tensor of one or more scalar fields Φ with potential $V(\Phi)$ whose kinetic energy term may contain a nontrivial metric on the scalar field manifold. The metric is assumed to be of the form

$$ds^2 = dr^2 + e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu, \quad (2.2)$$

where $\mu, \nu = 0, 1, 2, \ldots, d-2$ and $\eta_{\mu\nu}$ is the flat Minkowski metric. The scalar field is assumed to depend only on the transverse coordinate r, and if the prime denotes differentiation with respect to r then the Einstein equations require

$$-\Phi'\Phi' = (d-2)A'', \qquad \left(\frac{1}{2}\Phi'\Phi' - V\right) = \frac{(d-2)(d-1)}{2}(A')^2. \quad (2.3)$$

These two equations imply the scalar field equation,

$$\Phi'' + (d-1)\Phi'A' = \frac{\partial V}{\partial\Phi}. \quad (2.4)$$

If there is a nontrivial covariant metric on the scalar field manifold, the right-hand side of (2.4) includes the contravariant metric.

A domain wall solution separating two anti–de Sitter domains with the same cosmological constant would have $A \approx -|r|/a$ as $|r| \to \infty$.

If the potential V has the special form

$$V = \frac{1}{2}\left(\frac{\partial W}{\partial\Phi}\frac{\partial W}{\partial\Phi} - \frac{d-1}{d-2}W^2\right), \quad (2.5)$$

where $W = W(\Phi)$ is a suitable superpotential, and Einstein equations (2.3) and the scalar equation (2.4) are solved by solutions of the first order Bogomol'nyi equations:

$$\Phi' = \frac{\partial W}{\partial\Phi}, \qquad A' = -\frac{1}{d-2}W, \quad (2.6)$$

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Note that the spacetime is uniquely specified by giving a solution of (2.6) which is the same as the equation for a domain wall in the absence of gravity. One then obtains A by quadratures. The vacua correspond to critical points of the superpotential W. At these points the potential V is negative, and so one is in an anti–de Sitter phase. Recently, there has been a lot of interest in the possibility of obtaining such potentials within the context of $d = 5$ gauged supergravity models [4–7]. At present no superpotential with the correct properties derived from a supergravity model has yet been found. However, a solution was exhibited in [8] which is not derived from a supergravity model. We will return to this point in the last section. We will now show, without assuming that it is supersymmetric or satisfies the first order equations, how to superpose a smooth domain wall background with plane-fronted gravitational waves moving in the anti–de Sitter background.

3. pp-waves on the brane: the bound state.—An exact solution of Einstein's equations representing a gravitational wave moving at the speed of light in the x^1 direction is given by retaining the form $\Phi(r)$ and $A(r)$ but modifying the metric (2.2) to take the form

$$ds^2 = dr^2 + e^{2A(r)}[-du\,dv + H(u, r, x^i{}_\perp)du^2 + dx^i{}_\perp dx^i{}_\perp], \tag{3.1}$$

with $u = t - x^1$, $v = t + x^1$, $i = 2, \ldots, d - 3$, and where the u dependence of H is arbitrary but its dependence upon r and $x^i{}_\perp$ is governed by

$$H'' + (d - 1)H'A' + e^{-2A}\nabla^2{}_\perp H = 0\,, \tag{3.2}$$

where $\nabla^2{}_\perp$ is the flat Laplace operator in the coordinates $x^i{}_\perp$. This will have half as much supersymmetry as the domain wall background. One may further generalize this solution by replacing the flat metric $dx^i{}_\perp dx^i{}_\perp$ by an arbitrary $(d - 3)$-dimensional Ricci flat metric $g_\perp$. If $g_\perp$ admits covariantly constant spinors, then the background will still admit some supersymmetry.

If $g_\perp$ is flat space, solutions of (3.2) propagate in surfaces of constant r at the speed of light in the (arbitrarily chosen) x^1 direction with an amplitude depending upon r. Fourier analyzing in the $x_\perp$ direction gives $H \propto e^{ikx_\perp}$, where k could, in principle, depend upon u. If k is real, solutions would propagate faster than light in a given $r =$ const surface, and would appear as tachyons to an observer on the brane. On the other hand, solutions for which k is purely imaginary propagate on the brane like Kaluza-Klein modes. Thus, if $k^2 = -m^2$, i.e., $\nabla^2{}_\perp H = m^2 H$, we are led to the equation

$$H'' + (d - 1)H'A' + e^{-2A}m^2 H = 0\,. \tag{3.3}$$

Consider the zero modes, i.e., solutions with $m^2 = 0$. We take $H = F(r)H_{ij}(u)x^i{}_\perp x^j{}_\perp$ and find that $F = C_1 + C_2 \int^r ds e^{-(d-1)A(s)}$, where C_1 and C_2 are constants. The graviton perturbation $h = e^{-2A}H$ will diverge exponentially for large values of $|r|$ unless $C_2 = 0$. We will return to this divergence in the next section. The mode for which $C_2 = 0$, $C_1 = 1$, and

$$H = H_{ij}(u)x^i{}_\perp x^j{}_\perp \tag{3.4}$$

may be identified as a fully nonlinear version of the zero mode of Randall and Sundrum on a general domain wall background. Here, $H_{ij}(u)$ is an arbitrary trace free symmetric matrix which determines the polarization state of the graviton. The choice (3.4) is made so that the solution has a d-dimensional isometry group acting on the surfaces $r =$ const and $u =$ const. This invariance is not manifest in the coordinates $(r, u, v, x_\perp)$, but is in Rosen coordinates [9] $(\tilde{u}, \tilde{v}, \tilde{x}_\perp)$, in which (3.1), given (3.4), assumes the form

$$ds^2 = dr^2 + e^{2A}[-du\,d\tilde{v} + A_{ij}(u)d\tilde{x}^i{}_\perp d\tilde{x}^j{}_\perp], \tag{3.5}$$

where $u = \tilde{u}$, $v = \tilde{v} + \frac{1}{2}\dot{A}_{ij}(u)\tilde{x}^i{}_\perp \tilde{x}^j{}_\perp$, and $x^i{}_\perp = P^i{}_j(u)\tilde{x}^j{}_\perp$. Here, $A_{ij}(u) = P^m{}_i(u)P^m{}_j(u)$, the overdot denotes differentiation with respect to u, and the matrix $P^i{}_j(u)$ is a solution of $\ddot{P}^i{}_j = H_{ik}P^k{}_j$. To make contact with Refs. [1,2], we linearize, setting $P_{ij} = \delta_{ij} + \frac{1}{2}\psi_{ij}$, so that $\ddot{\psi}_{ij} = H_{ij}$. The quantity ψ is essentially the perturbation considered in [1]. Rosen coordinates are in general rather pathological at the nonlinear level and awkward to use. In our nonlinear analysis we shall, from now on, use only the coordinates $(r, u, v, x_\perp)$.

4. pp Wave in the bulk: blueshift and curvature singularities.—Our spacetimes are timelike and lightlike geodesically incomplete as $|r| \to \infty$. In the absence of gravitational waves, i.e., $H = 0$, $r = \infty$ corresponds to a regular Cauchy horizon, and the solution may be extended through the horizon (see, for example, Ref. [10]). If $H \neq 0$, however, the solutions will generically become singular as $|r| \to \infty$, and will not admit an extension. The nature of this singularity is most easily studied when the background is taken to be exactly AdS_d. If we let $z = ae^{r/a}$ then the metric (3.1) can be recast in so-called "Siklos" coordinates [11–14]:

$$ds^2 = \frac{a^2}{z^2}(dz^2 - du\,dv + H du^2 + dx^i{}_\perp dx^i{}_\perp), \tag{4.1}$$

where H now satisfies the generalized Siklos equation

$$z^{(d-2)}\frac{\partial}{\partial z}\left[\frac{1}{z^{d-2}}\frac{\partial H}{\partial z}\right] + \nabla^2{}_\perp H = 0\,.$$

Because all invariants formed from the Weyl tensor of (4.1) necessarily vanish, it is not possible to detect curvature singularities directly by calculating invariants. However, the necessary conditions that one may extend through the singularity in the metric at $z = \infty$ is that the components of the Riemann tensor in an orthonormal frame which has been parallel propagated along every timelike geodesic are finite. This requirement arises because freely falling observers move along timelike geodesics, and the components of the curvature tensor will measure the tidal forces which these observers experience. Following the demonstration in [14], one may calculate these terms explicitly for the Siklos metrics. One finds that certain frame components of the Riemann tensor generically assume the form

$$R_{(a)(b)(a)(b)} = \frac{\Lambda}{d-1} \pm z^5 \left(\frac{1}{z} \frac{\partial H}{\partial z} \right)_{,z}, \tag{4.2}$$

where $,z$ denotes differentiation relative to z, and where we have suppressed various constants which are irrelevant to this discussion. It follows that any solution with z dependence cannot be extended, and hence is singular. One sees that the z-dependent piece of (4.2) is the contribution from the Weyl tensor. It would therefore seem that the gravitons will be heavily "blueshifted" as we move towards large values of z.

If $\nabla^2{}_\perp H = m^2 H$, the Siklos equation has solutions of the form

$$H = z^{(d-1)/2} e^{ikx_\perp} [D_1 J_{(d-1)/2}(mz) + D_2 Y_{(d-1)/2}(mz)], \tag{4.3}$$

where $J_n(x)$ and $Y_n(x)$ are Bessel functions, and D_1, D_2 are some constants. The z dependence of H has the same form as the Kaluza-Klein modes of [1,2]. The behavior near $z = \infty$ shows that these are singular on the Cauchy horizon.

In order to get a better feel for the singular nature of these spacetimes, it is useful to focus on a specific example of a Siklos-type metric, where the z dependence is nontrivial. The simplest example is the higher-dimensional generalization [15,16] of *Kaigorodov's* spacetime [17], for which H is

$$H(z) = z^{d-1}.$$

The Kaigorodov metric is

$$ds^2 = \frac{a^2}{z^2} [-(1 - z^{d-1}) dt^2 - 2z^{d-1} dt dx^1 + (1 + z^{d-1})(dx^1)^2 + dz^2 + dx^2{}_\perp]. \tag{4.4}$$

This is the AdS_d analog of the simplest *vacuum pp* wave, namely, the homogeneous pp-wave in flat space. It has $d - 1$ obvious translational Killing vectors, and is also invariant under the $\mathbf{R}^+$ action:

$$(z, u, v) \rightarrow (\lambda z, \lambda^{(3-d)/2} u, \lambda^{(d+1)/2} v).$$

This action, combined with translations in u and v, generates a three-dimensional group of Bianchi type VI_h, where $h = \frac{-1}{(d-1)^{-2}}$. Therefore, the Kaigorodov isometry group contains a simply transitive subgroup which takes every point with z positive to any other point with z positive. A similar d-dimensional simply transitive group exists in the AdS_d case, for which the $\mathbf{R}^+$ action is simply $z \rightarrow \lambda z$. In the AdS_d case, we can extend beyond the reach of the group, in the Kaigorodov case we cannot.

Clearly, free falling timelike observers (who can cross the surface $z = \infty$ after a finite period of affine parameter time [14]) will see infinite tidal forces in this region. This shows that there are naked curvature singularities at the points $z = \infty$. Given our discussion in the previous section, where we saw that generic z-dependent graviton perturbations will diverge at large z, it is clear that we should regard these singularities as a generic feature of Siklos spacetimes.

5. Discussion.—We have shown that it is possible to include a nonlinear gravitational wave on a thick domain wall background, in such a way that one may recover the Randall-Sundrum bound state. Given the formal Witten style stability proofs in [4], which work as long as one has a solution of the first order equations, one might have thought that this would ensure that the Randall-Sundrum scenario could be perturbed in this way without problems. However, somewhat to our surprise, we have found that, generically, gravitons propagating in the bulk become singular on what is a Cauchy horizon in the unperturbed spacetime. These singularities are somewhat unusual, in that scalar invariants formed from the curvature tensor do not blow up but rather the components of the curvature in a parallel propagated frame along a timelike geodesic do blow up. Such singularities are called "pp curvature singularities" [14,18].

One might worry that these singularities signal a breakdown in our ability to make unitary predictions. However, any statements about unitarity should be restricted to physics *on the brane* at $z =$ const. Any pathological effects which may emerge from the singularity will be heavily redshifted by the time they reach the brane. Consequently, the extent to which these singularities signal a pathology of the theory is at present unclear. Interestingly, if one considers massless z-independent pp-waves (these would correspond to the Randall-Sundrum zero mode bound state), one finds that the components of the curvature do *not* blow up, and presumably the spacetime has a nonsingular extension.

In conclusion, we would like to return to the question of whether a suitable superpotential exists which can be derived from a supergravity model. The results of [4] and [7] show that, for the simplest case of a single scalar field in models of the type studied in [19], they do not. In fact, one may show quite generally that, for the models in [19] with an arbitrary number of scalar fields, they do not. The same is true for the models considered in [20]. It therefore remains an important open problem to find a suitable supergravity model or prove that no such model exists.

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