Exact Eigenstates and Magnetic Response of Spin-1 and Spin-2 Bose-Einstein Condensates

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The exact eigenspectra and eigenstates of spin-1 and spin-2 Bose-Einstein condensates (BECs) are found, and their response to a weak magnetic field is studied and compared with their mean-field counterparts. Whereas mean-field theory predicts the vanishing population of the zero magnetic-quantum-number component of a spin-1 antiferromagnetic BEC, the component is found to become populated as the magnetic field decreases. The spin-2 BEC exhibits an even richer magnetic response due to quantum correlations among three bosons.

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Bose-Einstein condensates (BECs) of alkali-metal atoms have internal degrees of freedom due to the hyperfine spin of the atoms. These degrees of freedom are frozen in a magnetic trap, but an optical trap liberates them to allow BEC to be in a superposition of magnetic sublevels [1]. BEC is therefore described by a vector rather than scalar order parameter. A new feature in this BEC system as compared to superfluid ^{3}He is the fact that its response to an external magnetic field is dominated by electronic rather than nuclear spin, and hence the response is much stronger than that of superfluid ^{3}He. This opens up possibilities of manipulating the magnetism of superfluid vapors. Observation of spin domains by an MIT group [2] offers a remarkable example of such manipulations. While the experiments reported so far achieved only the spin-1 BEC, the spin-2 BEC appears feasible by using the $F = 2$ multiplet of bosons such as ^{23}Na, ^{87}Rb, or ^{85}Rb.

The mean-field theory (MFT) for describing a vectorial BEC was developed by Ohmi and Machida [3] and by Ho [4] by generalizing the Gross-Pitaevskii equation under the restriction of gauge and spin-rotation symmetry; they also used it to predict various spin textures and topological excitations. Law *et al.* [5] utilized techniques developed in quantum optics [6,7] to study many-body states of spin-1 BEC in the absence of external fields, and found that spin-exchange collisions lead to rather complicated dynamical behavior of BEC that MFT fails to capture. In this Letter, we study magnetic response of spin-1 and spin-2 BECs by explicitly constructing exact eigenspectra and eigenstates, and compare the results with their mean-field counterparts.

We first consider a system of spin-1 identical bosons interacting via s-wave scattering. The second-quantized Hamiltonian of the bosons subject to a uniform magnetic field $\boldsymbol{B}$ and in a confining potential $U(\boldsymbol{r})$ is given by

$$\hat{H}_0 = \int d\mathbf{r}\bigg[\frac{\hbar^2}{2M}\nabla\hat{\Psi}_\alpha^\dagger\cdot\nabla\hat{\Psi}_\alpha + U\hat{\Psi}_\alpha^\dagger\hat{\Psi}_\alpha + \frac{\bar{c}_0}{2}\hat{\Psi}_\alpha^\dagger\hat{\Psi}_\beta^\dagger\hat{\Psi}_\beta\hat{\Psi}_\alpha + \frac{\bar{c}_1}{2}\hat{\Psi}_\alpha^\dagger\hat{\Psi}_\beta^\dagger\boldsymbol{f}_{\alpha\alpha'}\cdot\boldsymbol{f}_{\beta\beta'}\hat{\Psi}_{\beta'}\hat{\Psi}_{\alpha'} - g\mu_B\hat{\Psi}_\alpha^\dagger\boldsymbol{B}\cdot\boldsymbol{f}_{\alpha\alpha'}\hat{\Psi}_{\alpha'}\bigg], \tag{1}$$

where M is the mass of the bosons, $\hat{\Psi}_\alpha$ describes their field operator with magnetic quantum number $\alpha = -1, 0, 1$, and $\bar{c}_0$ and $\bar{c}_1$ are related to scattering lengths a_0 and a_2 of two colliding bosons with total angular momenta 0 and 2 by $\bar{c}_0 = 4\pi\hbar^2(2a_2 + a_0)/3M$ and $\bar{c}_1 = 4\pi\hbar^2(a_2 - a_0)/3M$ [4]. Here and henceforth, it is assumed that repeated indices are to be summed, and that the total number N of bosons in the system is fixed. We further assume that the external magnetic field is weak and $|c_1| \ll c_0$ so that the coordinate wave function $\phi(\boldsymbol{r})$ is independent of the spin state and solely determined by the first three terms of Eq. (1), namely, $[-\hbar^2\nabla^2/2M + U + \bar{c}_0(N-1)|\phi|^2]\phi = \epsilon\phi$. While spin domains were observed in the experiments subject to a gradient magnetic field [2] or in metastable states [8], here we assume a uniform magnetic field and do not consider the possibility of phase separation. Substituting $\hat{\Psi}_\alpha = \hat{a}_\alpha\phi$ into Eq. (1) and keeping only spin-dependent terms, we obtain

$$\hat{H} = (c_1/2) :\hat{\boldsymbol{F}}\cdot\hat{\boldsymbol{F}}: - p\hat{F}_z, \tag{2}$$

where $c_1 \equiv \bar{c}_1\int d\boldsymbol{r}|\phi|^4$, $:\hat{O}:$ arranges the operator $\hat{O}$ in normal order, and the three components $\hat{F}_{x,y,z}$ of the hyperfine-spin operator $\hat{\boldsymbol{F}}$ are defined in terms of 3×3 spin-1 matrices $F_{x,y,z}$ as $\hat{F}_x = (F_x)_{\alpha\beta}\hat{a}_\alpha^\dagger\hat{a}_\beta$, etc. In the following discussions we assume that $p \equiv g\mu_B B > 0$.

Exact energy eigenstates and eigenvalues of Hamiltonian (2) can be constructed as follows. We introduce an operator $\hat{A}^\dagger \equiv [(\hat{a}_0^\dagger)^2 - 2\hat{a}_1^\dagger\hat{a}_{-1}^\dagger]/\sqrt{3}$ which creates a pair of bosons in the spin-singlet state when operated on the vacuum, and define a set of states $|N_2, F, F_z\rangle$ as $|N_2, F, F_z\rangle \equiv Z^{-1/2}(\hat{A}^\dagger)^{N_2}(\hat{F}_-)^{F-F_z}(\hat{a}_1^\dagger)^F|\text{vac}\rangle$, where Z is the normalization constant and $\hat{F}_-$ is the lowering operator for F_z. Since $\hat{A}^\dagger$ commutes with $\hat{\boldsymbol{F}}^2$ and $\hat{F}_z$, $|N_2, F, F_z\rangle$ is the simultaneous eigenstate of $\hat{N}$, $\hat{\boldsymbol{F}}^2$, and $\hat{F}_z$, with total number of bosons $N = F + 2N_2$, total spin F, and magnetic quantum number F_z. This

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state is thus an energy eigenstate of $\hat{H}$ with energy eigenvalue

$$E = (c_1/2)[F(F + 1) - 2N] - pF_z. \quad (3)$$

The number of possible states $|N_2, F, F_z\rangle$ for a fixed N is obtained as the coefficient of x^N of the generating function $\sum_{N_2,F,F_z} x^N = \sum_{N_2,F}(2F + 1)x^{2N_2+F} = (1 - x)^{-3}$, and is given by $(N + 1)(N + 2)/2$. Since this number coincides with that of linearly independent states for a system of N spin-1 bosons, i.e., ${}_{N+2}C_2$, the set $\{|N_2, F, F_z\rangle\}$ forms a complete orthonormal basis.

The ground state is obtained by minimizing Eq. (3) with N held fixed. When $c_1 < 0$, it is a ferromagnetic state in which all bosons occupy the $m = 1$ state, in agreement with the prediction of MFT [2]. When $c_1 > 0$, $|N_2 = (N - F)/2, F, F_z = F\rangle$ is the exact ground state for

$$F - 1/2 < p/c_1 < F + 3/2. \quad (4)$$

That is, magnetization increases stepwise, taking the values $F = N - 2N_2$ with the step size $\Delta F = 2$ as the magnetic field increases. In contrast, MFT implies [2] that magnetization increases linearly with the magnetic field as $F_z \sim [g\mu_B/c_1]B$. Both theories, however, predict the same average slope. The difference between the exact ground state energy E and the minimum energy E_M in MFT is $E - E_M \sim -c_1(N - F)(2N + F)/2N$ [9], which is of the order of the antiferromagnetic interaction energy between one particular particle and the rest of the system.

From the form of the ground states $|N_2, F, F_z = F\rangle \propto (\hat{A}^\dagger)^{N_2}(\hat{a}_1^\dagger)^F|\text{vac}\rangle$, we may say that increasing the magnetic field breaks singlet "pairs" one by one, which results in the stepwise increase of magnetization. These pairs are in some sense analogous to Cooper pairs of electrons or ^{3}He, but there is a remarkable difference. In the case of Cooper pairs, the state is symmetric only under the permutations that do not break any pairs. On the other hand, in the present case the state is symmetric for any permutations of constituent particles.

An observable that makes a striking distinction from MFT is the population n_0 of the $m = 0$ Zeeman sublevel, which is predicted to be zero in MFT [2]. For the exact ground state, the expectation of n_0 is calculated to be

$$\bar{n}_0 \equiv \langle \hat{a}_0^\dagger \hat{a}_0 \rangle = \frac{2N_2}{2F + 3} = \frac{N - F}{2F + 3}. \quad (5)$$

The nonzero value of $\bar{n}_0$ makes a sharp distinction from the prediction of MFT. When $1 \ll F \ll N$, $\bar{n}_0$ is inversely proportional to the magnetic field as $\bar{n}_0 \sim \bar{c}_1\rho/(2g\mu_B B)$, where $\rho \equiv N\int d\boldsymbol{r}|\phi|^4$ is the average number density of BEC. Note that when $B \neq 0$, $\bar{n}_0/N$ is finite only in the mesoscopic regime, and vanishes in the limit $N \to \infty$. For sodium atoms in the $F = 1$ state, where $\bar{c}_1/\mu_B \sim 10^{-19}$ cm^3 G, experiments of Ref. [1] achieved $\rho \sim 10^{15}$ cm^{-3} in an optical dipole trap. From these values, we see, for example, that in order to observe $\bar{n}_0$ of the order of 10^3, a small magnetic field of the order of 10^{-7} G is required.

The rapid decrease of $\bar{n}_0$ as a function of F can be ascribed to the indistinguishability of bosons. If all particles were distinguishable, the state could be written as $\Psi = \prod_{i=1}^{N_2} \Psi_{2i-1,2i} \prod_{j=1}^{N-2N_2} |1\rangle_{2N_2+j}$, where $\Psi_{i,j}$ is the spin-singlet state for particles i and j. There are N_2 singlet pairs so that the $m = 0$ population would be $(N - F)/3$, which decreases only linearly with F. The wave function of a Bose system is obtained by the symmetrization of Ψ. Adding bosons in the $m = 1$ state increases the relative probability amplitudes having large $m = 1$ occupation numbers. This is nothing but the bosonic enhancement and may be interpreted as a consequence of the constructive interference among the permuted terms. The expectation value of the $m = 0$ population thus decreases rapidly towards the MFT value of zero with the increasing magnetic field, as can be seen from Eq. (5).

We next consider BEC of spin-2 bosons. Bose symmetry requires that the total angular momentum of two colliding bosons is restricted to 0, 2, and 4, so that the interaction Hamiltonian which describes binary collisions via the s-wave scattering is generally written as $\hat{V} = g_4\hat{P}_4 + g_2\hat{P}_2 + g_0\hat{P}_0$, where $\hat{P}_F(F = 0, 2, 4)$ denotes the projection operator for the total angular momentum F [4], g_F is related to scattering length a_F by $g_F = 4\pi\hbar^2 a_F/M$, and we have omitted the coordinate delta function describing the short-range nature of the interaction. Using $\hat{P}_4 + \hat{P}_2 + \hat{P}_0 = \hat{1}$ and $\hat{\boldsymbol{f}}_i \cdot \hat{\boldsymbol{f}}_j = 4\hat{P}_4 - 3\hat{P}_2 - 6\hat{P}_0$ (i and j label particles), $\hat{V}$ is rewritten as $\hat{V} = \bar{c}_0 + \bar{c}_1\hat{\boldsymbol{f}}_i \cdot \hat{\boldsymbol{f}}_j + \bar{c}_2\hat{P}_0$, where $\bar{c}_0 = (3g_4 + 4g_2)/7$, $\bar{c}_1 = (g_4 - g_2)/7$, and $\bar{c}_2 = (3g_4 - 10g_2 + 7g_0)/7$.

To derive the second-quantized form of the Hamiltonian, it is convenient to introduce a new operator $\hat{S}_+ \equiv (\hat{a}_0^\dagger)^2/2 - \hat{a}_1^\dagger\hat{a}_{-1}^\dagger + \hat{a}_2^\dagger\hat{a}_{-2}^\dagger$. This operator creates, if applied to the vacuum, a pair of bosons in the spin-singlet state. This pair, however, should not be regarded as a single composite boson because $\hat{S}_+$ does not satisfy the Bose commutation relations. The operator $\hat{S}_+$ instead satisfies the SU(1, 1) commutation relations if we define $\hat{S}_- \equiv \hat{S}_+^\dagger$ and $\hat{S}_z \equiv (2\hat{N} + 5)/4$, namely, $[\hat{S}_z, \hat{S}_\pm] = \pm\hat{S}_\pm$ and $[\hat{S}_+, \hat{S}_-] = -2\hat{S}_z$; the minus sign in the last equation is the only distinction from the usual spin commutation relations. Accordingly, the Casimir operator $\hat{S}^2$ that commutes with $\hat{S}_\pm$ and $\hat{S}_z$ should be defined as $\hat{S}^2 \equiv -\hat{S}_+\hat{S}_- + \hat{S}_z^2 - \hat{S}_z$. The requirement that $\hat{S}_z^2 - \hat{S}_z - \hat{S}^2$ must be positive semidefinite leads to the allowed combinations of eigenvalues $\{S(S - 1), S_z\}$ for $\hat{S}^2$ and $\hat{S}_z$ such that $S = (2N_0 + 5)/4$ ($N_0 = 0, 1, 2, \ldots$) and $S_z = S + N_2$ ($N_2 = 0, 1, 2, \ldots$). Here we introduced new quantum numbers N_2 and N_0, where the operator $\hat{S}_+$ raises N_2 by one and the relation $N = 2N_2 + N_0$ holds. We may thus interpret N_2 as the number of spin-singlet pairs, and N_0 as that of the remaining bosons. Noting that the second-quantized form of $\hat{P}_0$ is written as $2\hat{S}_+\hat{S}_-/5$, the second-quantized form of the spin-dependent part of the Hamiltonian can be written as

$$\hat{H} = (c_1/2) : \hat{\boldsymbol{F}} \cdot \hat{\boldsymbol{F}} : +(2c_2/5)\hat{S}_+\hat{S}_- - p\hat{F}_z\,, \quad (6)$$

where $c_i \equiv \bar{c}_i \int d\boldsymbol{r}|\phi|^4$.

We first discuss MFT with a fixed total number of bosons, and define a state in which all bosons are in the same single-particle state as $(N!)^{-1/2}(\sum_\alpha \zeta_\alpha \hat{a}_\alpha^\dagger)^N|\text{vac}\rangle$, where $\sum_\alpha |\zeta_\alpha|^2 = 1$. Noting that $\langle \hat{a}_{\alpha'}^\dagger \hat{a}_{\beta'}^\dagger \hat{a}_\beta \hat{a}_\alpha\rangle = N(N-1)\zeta_{\alpha'}^*\zeta_{\beta'}^*\zeta_\beta\zeta_\alpha$, the energy E_M of the state is written as

$$E_M = N(N-1)\left[\left(\frac{c_1}{2}\right)\langle\hat{\boldsymbol{f}}\rangle^2 + \left(\frac{2c_2}{5}\right)s^2\right] - Np\langle\hat{f}_z\rangle\,, \quad (7)$$

where $\langle\hat{\boldsymbol{f}}\rangle^2 = \langle\hat{f}_z\rangle^2 + |2(\zeta_2\zeta_1^* + \zeta_{-1}\zeta_{-2}^*) + \sqrt{6}\,\zeta_1\zeta_0^* + \zeta_0\zeta_{-1}^*|^2$, $\langle\hat{f}_z\rangle = 2(|\zeta_2|^2 - |\zeta_{-2}|^2) + |\zeta_1|^2 - |\zeta_{-1}|^2$, and $s^2 \equiv |\zeta_0^2/2 - \zeta_1\zeta_{-1} + \zeta_2\zeta_{-2}|^2$ [9]. [An MFT that assumes coherent states with amplitudes $\{\sqrt{N}\,\zeta_\alpha\}$ for the ground state is obtained by replacing the terms $c_i(N-1)$ in E_M by c_iN.] The ground state and its magnetization in MFT are obtained by minimizing E_M, and our results are summarized as follows. When $c_2 > 0$ and $c_1 > 0$, the term including c_2 vanishes ($s^2 = 0$) for the minimized state, and the magnetization increases linearly with the magnetic field as $F_z \sim [g\mu_B/c_1]B$. Any Zeeman sublevel can take nonzero population in this case. When $c_2 < 0$ and $20c_1 + |c_2| > 0$, the c_2 term contributes to F_z, but it amounts only to replacing c_1 in the expression of F_z above with $c_1 + |c_2|/20$. This case is quite similar to the spin-1 case, and only $m = \pm 2$ levels are populated. In other regions of the parameters c_1 and c_2, the ground state is ferromagnetic.

Exact energy eigenstates and eigenvalues of Hamiltonian (6) can be obtained as follows. Because operators $\hat{S}_\pm$ are invariant under any rotation of the system, they commute with $\hat{\boldsymbol{F}}^2$ and $\hat{F}_z$. The energy eigenstates can thus be classified according to quantum numbers N_2 and N_0, total spin F, and magnetic quantum number F_z. We thus denote the eigenstates as $|N_2, N_0, F, F_z, \lambda\rangle$, where $\lambda = 1, 2, \ldots, g_{N_0,F}$ is included to label degenerate states. The energy eigenvalue for this state is

$$E = (c_1/2)[F(F+1) - 6N] + (c_2/10)(N - N_0) \times (N + N_0 + 3) - pF_z\,, \quad (8)$$

where we used $2N_2 + N_0 = N$. The degeneracy $g_{N_0,F}$ can be calculated from generating function [9]

$$\sum_{N_0,F} g_{N_0,F}x^{N_0}y^F = \frac{1 - xy + x^2y^2}{(1 - xy^2)(1 + xy)(1 - x^3)}\,. \quad (9)$$

The total spin F can take integer values in the range $0 \le F \le 2N_0$ except for some forbidden values. That is, $F = 1, 2, 5, 2N_0 - 1$ is not allowed when $N_0 = 3k$ ($k \in \boldsymbol{Z}$), and $F = 0, 1, 3, 2N_0 - 1$ is forbidden when $N_0 = 3k \pm 1$.

It is convenient to consider two cases separately depending on the sign of the parameter c_2 for the minimization of Eq. (8).

(a) $c_2 > 0$.—The ground state is $|N_2 = 0, N_0 = N, F, F_z = F, \lambda\rangle$ with F taking the value closest to $p/c_1 - 1/2$. The magnetization $F_z = F$ can take integer values in the range $0 \le F \le 2N$ except for the forbidden values described above.

The spin correlations in these ground states are rather complicated in comparison with the spin-1 case. This is because the condition $\langle\hat{S}_+\hat{S}_-\rangle = 0$ implies that the spin correlation between *any* two particles must avoid the singletlike correlation. Except for this constraint, the spin correlation may be reduced to a combination of two- and three-particle correlations. Let us define the operator $\hat{A}_F^{(n)\dagger}$ such that it creates n bosons in the state with total spin F and $F_z = F$ when applied to the vacuum. Consider a set of unnormalized states,

$$|n_{12}, n_{22}, n_{30}, n_{33}\rangle = \hat{\hat{P}}_0(\hat{a}_2^\dagger)^{n_{12}}(\hat{A}_2^{(2)\dagger})^{n_{22}}(\hat{A}_0^{(3)\dagger})^{n_{30}} \times (\hat{A}_3^{(3)\dagger})^{n_{33}}|\text{vac}\rangle\,, \quad (10)$$

with $n_{12}, n_{22}, n_{30} = 0, 1, 2, \ldots,$ and $n_{33} = 0, 1$. The operator $\hat{\hat{P}}_0$ is the projection to the kernel of $\hat{S}_-$, which ensures $\langle\hat{S}_+\hat{S}_-\rangle = 0$ for these states. It is easy to see that $|n_{12}, n_{22}, n_{30}, n_{33}\rangle$ are energy eigenstates with $N_2 = 0$, $N_0 = n_{12} + 2n_{22} + 3n_{30} + 3n_{33}$, and $F = F_z = 2n_{12} + 2n_{22} + 3n_{33}$. Note that the states belonging to the same eigenvalue are not necessarily mutually orthogonal. A further analysis [9], however, shows that these states are linearly independent, and the degeneracy coincides with $g_{N_0,F}$. The set (10) thus forms a complete basis of the subspace spanned by $\{|N_2 = 0, N_0, F, F_z = F, \lambda\rangle\}$. The form in (10) provides an intuitive explanation for the forbidden values of F. For example, $F = 0$ is possible only when N_0 is a multiple of 3 because the singlet state is formed by only three particles.

(b) $c_2 < 0$.—The ground state should satisfy $F_z = F$. To determine the remaining parameters $\{N_0, F\}$, we first separate E into the part that depends on F and $l \equiv 2N_0 - F$, and the part that depends only on N, namely,

$$E = \frac{(20c_1 + |c_2|)}{40}g(F, l) - 3c_1N - \frac{|c_2|}{10}N(N+3)\,, \quad (11)$$

where $g(F, l) = F^2 + [1 + c(5 + 2l) - p']F + cl(l + 6)$, $c \equiv |c_2|/(20c_1 + |c_2|)$, and $p' \equiv 40p/(20c_1 + |c_2|)$.

When $20c_1 + |c_2| < 0$, the ground state is obtained by maximizing $g(F, l)$. Suppose first that N is even. Since $g(F, l)$ is a decreasing function of l in this case, the maximum should be either $g(0, 0) = 0$ or $g(2N, 0) = 2N(2N + 1 + 5c - p')$. The ground state is thus $\{N_0, F\} = \{0, 0\}$ if $40p < 5|c_2| - (2N + 1)\,|20c_1 + |c_2\,||$, and $\{N_0, F\} = \{N, 2N\}$ otherwise. When N is odd, $\{N_0, F\} = \{0, 0\}$ is not allowed, and we must compare $g(0, 6)$, $g(2, 0)$, and $g(2N, 0)$. The ground state is $\{N_0, F\} = \{1, 2\}$ if $40p < 5|c_2| - (2N + 3)\,|20c_1 + |c_2\,||$, and

$\{N_0, F\} = \{N, 2N\}$ otherwise. These results indicate that in the small parameter region of $-5|c_2|/2N \lesssim 20c_1 + |c_2| < 0$, magnetization of the ground state jumps from 0 or 2 to $2N$. Such a large discontinuity does not appear in MFT with a linear Zeeman potential. (However, in the presence of a quadratic Zeeman potential, such a jump occurs also in MFT [2].)

When $20c_1 + |c_2| > 0$, the ground state is obtained by minimizing $g(F, l)$. The function $g(x, 0)$ for real x is minimal when $x = x_0 \equiv (p' - 5c - 1)/2$. Since $l = 0$ is allowed only when $F = k' \equiv 2N - 4k$ with k being a non-negative integer, it is sufficient to compare the states with $\{N_0, F\} = \{k'/2 - 2, k' - 4\}, \{k'/2, k' - 3\}, \{k'/2, k' - 2\}, \{k'/2 + 2, k' - 1\}, \{k'/2, k'\}$ when x_0 is in the region $[k' - 4, k']$. The ground state is thus $\{N_0, F\} = \{k'/2, k'\}$ if

$$\max\{-1 - c(2k' - 1), -10c(k' + 4)\} < p' - 2k' < \min\{2 + 2c(3k' + 19), 3 + c(2k' + 17)\}, \tag{12}$$

$\{N_0, F\} = \{k'/2 + 2, k' - 1\}$ if

$$-2 + 6c(k' + 7) < p' - 2k' < -10c(k' + 4), \tag{13}$$

$\{N_0, F\} = \{k'/2, k' - 2\}$ if

$$\max\{-5 + c(2k' + 9), -4 - 2c(k' - 2)\} < p' - 2k' < \min\{-2 + 6c(k' + 7), -1 - c(2k' - 1)\}, \tag{14}$$

and $\{N_0, F\} = \{k'/2, k' - 3\}$ if

$$-6 + 2c(3k' + 7) < p' - 2k' < -4 - 2c(k' - 2). \tag{15}$$

These results indicate how magnetization increases with the applied magnetic field. When $F_z \lesssim 1/8c$, F_z takes all integer values. In the region $1/8c \lesssim F_z \lesssim 1/4c$, F_z skips the values $F_z = 2N - 4k - 1$. When $1/4c \lesssim F_z \lesssim 1/c$, $F_z = 2N - 4k - 3$ are further skipped, and F_z takes every other integer values. When $1/c \lesssim F_z$, $F_z = 2N - 4k$ are the only allowed values, so F_z increases by 4 at a time.

The reduced form of the states mentioned above is also helpful to illustrate this behavior. The states with $F = k' - 4, k' - 3, k' - 2, k' - 1, k'$ can be written as $(\hat{A}_0^{(2)\dagger})^{k+1}(\hat{a}_2^\dagger)^{k'/2-2}|\text{vac}\rangle$, $(\hat{A}_0^{(2)\dagger})^{k}(\hat{a}_2^\dagger)^{k'/2-3}\hat{A}_3^{(3)\dagger}|\text{vac}\rangle$, $(\hat{A}_0^{(2)\dagger})^{k}(\hat{a}_2^\dagger)^{k'/2-2}\hat{A}_2^{(2)\dagger}|\text{vac}\rangle$, $(\hat{A}_0^{(2)\dagger})^{k-1}(\hat{a}_2^\dagger)^{k'/2-3}\hat{A}_2^{(2)\dagger}\hat{A}_3^{(3)\dagger}|\text{vac}\rangle$, $(\hat{A}_0^{(2)\dagger})^{k}(\hat{a}_2^\dagger)^{k'/2}|\text{vac}\rangle$, respectively. As the energy cost required to break a singlet pair increases, the transitions accompanied by this breakage require a stronger field and are eventually suppressed.

Contrary to MFT, the exact ground state shows nonzero population in the $m = 0, \pm 1$ levels. Since the expressions of the exact results for these populations are lengthy, we show only the leading terms under the condition $1 \ll n_{12} \ll N_2$. Surprisingly, the populations are considerably different for the types of possible ground states, namely, $(\hat{A}_0^{(2)\dagger})^{N_2}(\hat{a}_2^\dagger)^{n_{12}}(\hat{A}_2^{(2)\dagger})^{n_{22}}(\hat{A}_3^{(3)\dagger})^{n_{33}}|\text{vac}\rangle$ with $n_{22} = 0, 1$ and $n_{33} = 0, 1$. The results are $\langle \hat{a}_1^\dagger \hat{a}_1 \rangle \sim \langle \hat{a}_{-1}^\dagger \hat{a}_{-1} \rangle \sim N_2(1 + n_{33})/n_{12}$ and $\langle \hat{a}_0^\dagger \hat{a}_0 \rangle \sim N_2(1 + 2n_{22})/n_{12}$. These results indicate that the populations in the $m = 0, \pm 1$ states are very sensitive to the combination of the spin correlations, and a very small difference in magnetization leads to large changes in the populations, by a factor of 2 or 3. The origin of this drastic change is the bosonic enhancement caused by the term $(\hat{a}_2^\dagger)^2 \hat{a}_{-1}^\dagger$ in $\hat{A}_3^{(3)\dagger}$ and the term $\hat{a}_2^\dagger \hat{a}_0^\dagger$ in $\hat{A}_2^{(2)\dagger}$.

To summarize, we examined magnetic response of spin-1 and spin-2 BECs by deriving exact eigenstates of each Hamiltonian. The response is stepwise and the spin-1 BEC shows the step of 2 units reflecting formation or destruction of singletlike pairs. In the spin-2 case, the spin correlations among three particles appear, leading to various step sizes ranging from 1 to 4 units. In a small parameter region, magnetization jumps from almost zero to the maximum of the order of N. Some Zeeman-level populations, which are predicted to be zero in MFT, are found to be nonzero when the magnetic field is small. These populations decrease rapidly with the increasing magnetic field, which can be understood as a consequence of bosonic enhancement. The bosonic enhancement also serves as an "amplifier" of a small change in spin correlations because it leads to large changes in Zeeman-level populations in the spin-2 BEC.

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