# Radiative Tail of Realistic Rotating Gravitational Collapse

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An astrophysically realistic model of wave dynamics in black-hole spacetimes must involve a nonspherical background geometry with *angular momentum*. We consider the evolution of gravitational (and electromagnetic) perturbations in rotating Kerr spacetimes. We show that a rotating Kerr black hole becomes "bald" slower than the corresponding spherically symmetric Schwarzschild black hole. Moreover, our results turn over the traditional belief (which has been widely accepted during the last three decades) that the late-time tail of gravitational collapse is universal. Our results are also of importance both to the study of the no-hair conjecture and the mass-inflation scenario (stability of Cauchy horizons).

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The radiative tail of gravitational collapse decays with time leaving behind a Kerr-Newman black hole characterized solely by the black-hole mass, charge, and angular momentum. This is the essence of the no-hair conjecture, introduced by Wheeler in the early 1970s [1].

Price [2] was the first to analyze the mechanism by which the spacetime outside a (nearly spherical) star divests itself of all radiative multipole moments, and leaves behind a Schwarzschild black hole; it was demonstrated that all radiative perturbations decay asymptotically as an inverse power of time, the power indices equal $2l + 3$ (in absolute value), where $l$ is the multipole order of the perturbation. This late-time decay of radiative fields is often referred to as their "power-law tail." Physically, these inverse power-law tails are associated with the backscattering of waves off the effective curvature potential at asymptotically far regions [2,3].

The analysis of Price has been extended by many authors. We shall not attempt to review the numerous works which address the problem of the late-time evolution of gravitational collapse. For a partial list of references, see, e.g., [4]. These earlier analyses were restricted, however, to spherically symmetric backgrounds. It is well known that realistic stellar objects generally rotate about their axis, and are therefore not spherical. Thus, the nature of the physical process of stellar core collapse to form a black hole is essentially nonspheric. An astrophysically realistic model must therefore take into account the angular momentum of the background geometry.

The corresponding problem of wave dynamics in realistic rotating Kerr spacetimes is much more complicated due to the lack of spherical symmetry. A first progress has been achieved only recently [5–8]. Detailed analyses for the simplified toy model of a test scalar field in the Kerr background have been given recently in [9,10].

Obviously, the most interesting situation from a physical point of view is the dynamics of gravitational perturbations in rotating Kerr spacetimes. This is the subject of this Letter, in which we present our main results for this fascinating problem. Full details of the analysis are given elsewhere [4].

The dynamics of massless perturbations outside a realistic rotating Kerr black hole is governed by Teukolsky's master equation [11,12]:

$$\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\varphi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\varphi^2} - \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\times \left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - 2s\left[\frac{a(r-m)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi}{\partial\varphi} - 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\psi}{\partial t} + (s^2\cot^2\theta - s)\psi = 0, \tag{1}$$

where $M$ and $a$ are the mass and angular momentum per unit mass of the black hole, and $\Delta = r^2 - 2Mr + a^2$. (We use gravitational units in which $G = c = 1$.) The parameter $s$ is called the spin weight of the field. For gravitational perturbations $s = \pm 2$ (for electromagnetic perturbations $s = \pm 1$). The field quantities $\psi$ which satisfy Teukolsky's equation are given in [12].

Resolving the field in the form $\psi = \Delta^{-s/2}(r^2 + a^2)^{-1/2}\sum_{m=-\infty}^{\infty}\Psi^m e^{im\varphi}$ (where $m$ is the azimuthal number), one obtains a wave equation for each value of $m$

$$D\Psi \equiv \left[B_1\frac{\partial^2}{\partial t^2} + B_2\frac{\partial}{\partial t} - \frac{\partial^2}{\partial y^2} + B_3 - \frac{\Delta}{(r^2+a^2)^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right]\Psi = 0, \tag{2}$$

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where the tortoise radial coordinate $y$ is defined by $dy = [(r^2 + a^2)/\Delta]dr$. (We suppress the index $m$.) The coefficients $B_i = B_i(r, \theta)$ are given by

$$B_1(r, \theta) = 1 - \frac{\Delta a^2 \sin^2\theta}{(r^2 + a^2)^2}, \quad (3)$$

and

$$B_2(r, \theta) = \left\{\frac{4iMmar}{\Delta} - 2s\left[\frac{M(r^2 - a^2)}{\Delta} - r - ia\cos\theta\right]\right\}\frac{\Delta}{(r^2 + a^2)^2}. \quad (4)$$

[The explicit expression of $B_3(r, \theta)$ is not important for the analysis.]

The time evolution of a wave field described by Eq. (2) is given by

$$\Psi(z, t) = 2\pi \int \int_0^{\pi} \{B_1(z')[G(z, z'; t)\Psi_t(z', 0) + G_t(z, z'; t)\Psi(z', 0)] + B_2(z')G(z, z'; t)\Psi(z', 0)\} \sin\theta' \, d\theta' \, dy', \quad (5)$$

for $t > 0$, where $z$ stands for $(y, \theta)$. The (retarded) Green's function $G(z, z'; t)$ is defined by $DG(z, z'; t) = \delta(t)\delta(y - y')\delta(\theta - \theta')/2\pi \sin\theta$ with $G = 0$ for $t < 0$. We express the Green's function in terms of the Fourier transform $\tilde{G}_l(y, y'; w)$

$$G(z, z'; t) = \frac{1}{(2\pi)^2} \sum_{l=l_0}^{\infty} \int_{-\infty+ic}^{\infty+ic} \tilde{G}_l(y, y'; w)_s S_l^m(\theta, aw)_s S_l^m(\theta', aw) e^{-iwt} \, dw, \quad (6)$$

where $c$ is some positive constant and $l_0 = \max(|m|, |s|)$. The functions ${}_sS_l^m(\theta, aw)$ are the spin-weighted spheroidal harmonics which are solutions to the angular equation [12]

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \left(a^2w^2\cos^2\theta - \frac{m^2}{\sin^2\theta} - 2aws\cos\theta - \frac{2ms\cos\theta}{\sin^2\theta} - s^2\cot^2\theta + s + {}_sA_l^m\right){}_sS_l^m = 0. \quad (7)$$

The Fourier transform is analytic in the upper half $w$ plane and it satisfies the equation [12]

$$\tilde{D}(w)\tilde{G}_l \equiv \left\{\frac{d^2}{dy^2} + \left[\frac{K^2 - 2is(r - M)K + \Delta(4irws - \lambda)}{(r^2 + a^2)^2} - H^2 - \frac{dH}{dy}\right]\right\}\tilde{G}_l(y, y'; w) = \delta(y - y'), \quad (8)$$

where $K = (r^2 + a^2)w - am$, $\lambda = A + a^2w^2 - 2amw$, and $H = s(r - M)/(r^2 + a^2) + r\Delta/(r^2 + a^2)^2$.

Define two auxiliary functions $\tilde{\Psi}_1$ and $\tilde{\Psi}_2$ as solutions to the homogeneous equation $\tilde{D}(w)\tilde{\Psi}_1 = \tilde{D}(w)\tilde{\Psi}_2 = 0$, with the physical boundary conditions of purely ingoing waves crossing the event horizon, and purely outgoing waves at spatial infinity, respectively. In terms of $\tilde{\Psi}_1$ and $\tilde{\Psi}_2$, and henceforth assuming $y' < y$, $\tilde{G}_l(y, y'; w) = -\tilde{\Psi}_1(y', w)\tilde{\Psi}_2(y, w)/W(w)$, where we have used the Wronskian relation $W(w) = W(\tilde{\Psi}_1, \tilde{\Psi}_2) = \tilde{\Psi}_1\tilde{\Psi}_{2,y} - \tilde{\Psi}_2\tilde{\Psi}_{1,y}$.

It is well known that the late-time behavior of massless perturbation fields is determined by the backscattering from asymptotically far regions [2,3]. Thus, the late-time behavior is dominated by the low-frequencies contribution to the Green's function, for only low frequencies will be backscattered by the small effective curvature potential (at $r \gg M$). Therefore, a small-$w$ approximation [or equivalently, a large-$r$ approximation of Eq. (8)] is sufficient in order to study the asymptotic late-time behavior of the fields [13]. With this approximation, the two basic solutions required in order to build the Fourier transform are $\tilde{\Psi}_1 = r^{l+1}e^{iwr}M(l + s + 1 - 2iwM, 2l + 2, -2iwr)$ and $\tilde{\Psi}_2 = r^{l+1}e^{iwr}U(l + s + 1 - 2iwM, 2l + 2, -2iwr)$, where $M(a, b, z)$ and $U(a, b, z)$ are the two standard solutions to the confluent hypergeometric equation [14]. Then $W(\tilde{\Psi}_1, \tilde{\Psi}_2) = i(-1)^{l+1}(2l + 1)!\,(2w)^{-(2l+1)}/(l + s)!$.

In order to calculate $G(z, z'; t)$ using Eq. (6), one may close the contour of integration into the lower half of the complex frequency plane. Then, one identifies three distinct contributions to $G(z, z'; t)$ [15]: prompt contribution, quasinormal modes, and tail contribution. The late-time tail is associated with the existence of a branch cut (in $\tilde{\Psi}_2$) in the complex frequency plane [15] (usually placed along the negative imaginary $w$-axis). A little arithmetic leads to [16]

$$\tilde{G}_l^C(y, y'; w) = \left[\frac{\tilde{\Psi}_2(y, we^{2\pi i})}{W(we^{2\pi i})} - \frac{\tilde{\Psi}_2(y, w)}{W(w)}\right]\tilde{\Psi}_1(y', w) = \frac{(-1)^{l-s}4\pi Mw(l - s)!}{(2l + 1)!}\frac{\tilde{\Psi}_1(y, w)\tilde{\Psi}_1(y', w)}{W(w)}. \quad (9)$$

Taking cognizance of Eq. (6) we obtain

$$G^C(z, z'; t) = \sum_{l=l_0}^{\infty} \frac{iM(-1)^s 2^{2l+1}(l + s)!\,(l - s)!}{\pi[(2l + 1)!]^2} \int_0^{-i\infty} \tilde{\Psi}_1(y, w)\tilde{\Psi}_1(y', w)_s S_l(\theta, aw)_s S_l(\theta', aw) w^{2l+2} e^{-iwt} \, dw. \quad (10)$$

The angular equation (7) is amenable to a perturbation treatment for small $aw$; we write it in the form $(L^0 + L^1)_s S_l^m = -_s A_l^m {}_s S_l^m$, where $L^1(\theta, aw) \equiv (aw)^2 \cos^2\theta - 2aws\cos\theta$ [and $L^0(\theta)$ is the $w$-independent part of Eq. (7)], and we use the spin-weighted spherical functions ${}_sY_l^m$ as a representation. They satisfy $L^0 {}_sY_l = -{}_sA_l^{(0)} {}_sY_l$ with ${}_sA_l^{(0)} = (l - s)(l + s + 1)$. For small $aw$ a standard perturbation theory (see, for example, [17]) yields [4]

$$ {}_sS_l(\theta, aw) = \sum_{k=l_0}^{\infty} C_{lk}(aw)^{|l-k|} {}_sY_k(\theta), \tag{11} $$

where, to leading order in $aw$, the coefficients $C_{lk}(aw)$ are $w$ independent [4]. Equation (11) implies that the black-hole rotation mixes different spin-weighted spherical harmonics.

The time evolution of the fields is given by Eq. (5). Therefore, in order to elucidate the coupling between different modes we should evaluate the integrals $\langle slm \mid skm\rangle$, $\langle slm| \sin^2\theta |skm\rangle$, and $\langle slm| \cos\theta |skm\rangle$, where $\langle slm|F(\theta)|skm\rangle \equiv \int {}_sY_l^{m*} F(\theta) {}_sY_k^m \, d\Omega$ [see Eqs. (3) and (4) for the definition of the $B_i(r,\theta)$ coefficients]. The spin-weighted spherical harmonics are related to the rotation matrix elements of quantum mechanics [18]. Hence, standard formulas are available for integrating the product of three such functions (these are given in terms of the Clebsch-Gordan coefficients [4]). In particular, the integral $\langle sl0| \sin^2\theta |sk0\rangle$ vanishes unless $l = k, k \pm 2$, while the integral $\langle sl0| \cos\theta |sk0\rangle$ vanishes unless $l = k \pm 1$. For nonaxially symmetric ($m \neq 0$) modes, $\langle slm| \sin^2\theta |skm\rangle \neq 0$ for $l = k, k \pm 1, k \pm 2$, and $\langle slm| \cos\theta |skm\rangle \neq 0$ for $l = k, k \pm 1$ (all other matrix elements vanish).

*Asymptotic behavior at timelike infinity.*—As already explained, the late-time behavior of the fields should follow from the low-frequency contribution to the Green's function. Actually, it is easy to verify that the effective contribution to the integral in Eq. (10) should come from $|w| = O(1/t)$. Thus, in order to obtain the asymptotic behavior of the fields at timelike infinity (where $y, y' \ll t$) we may use the $|w|r \ll 1$ asymptotic limit of $\tilde{\Psi}_1(r,w)$, which is given by $\tilde{\Psi}_1(r,w) \simeq r^{l+1}$ [14].

Substituting this in Eq. (10), and using the representation Eq. (11) for the spin-weighted spheroidal wave functions ${}_sS_l$ [together with the above cited properties of the integrals $\langle slm| \sin^2\theta |skm\rangle$ and $\langle slm| \cos\theta |skm\rangle$], we find that the asymptotic late-time behavior of the $l$ mode (where $l \geq l_0$) is dominated by the following effective Green's function:

$$ G_l^C(z, z'; t) = MF_1(yy')^{l_0+1} {}_sY_l(\theta) {}_sY_{l_0}^*(\theta') a^{l-l_0} t^{-(l+l_0+3)}, \tag{12} $$

where $F_1 = F_1(l, l_0, m, s) = (-1)^{(l+l_0+2s+2)/2} 2^{2l_0+1}(l + l_0 + 2)!\,(l_0 + s)!\,(l_0 - s)!\, C_{l_0 l}/\pi[(2l_0 + 1)!]^2$. We emphasize that the power indices $l + l_0 + 3$ found here for rotating Kerr spacetimes are smaller than the corresponding power indices (the well known $2l + 3$) in spherically symmetric Schwarzschild spacetimes. (There is an equality only for the $l = l_0$ mode.) This implies a slower decay of perturbations in rotating Kerr spacetimes.

*Asymptotic behavior at future null infinity.*—It is easy to verify that for this case the effective frequencies contributing to the integral in Eq. (10) are of order $O(1/u)$. Thus, for $y - y' \ll t \ll 2y - y'$ one may use the $|w|y' \ll 1$ limit for $\tilde{\Psi}_1(y', w)$ and the $M \ll |w|^{-1} \ll y$ ($\mathrm{Im} w < 0$) asymptotic limit of $\tilde{\Psi}_1(y, w)$, which is given by $\tilde{\Psi}_1(y, w) \simeq e^{iwy}(2l + 1)!\, e^{-i\pi(l+s+1)/2} \times (2w)^{-(l+s+1)} y^{-s}/(l - s)!$ [14].

Substituting this in Eq. (10), and using the representation Eq. (11) for the spin-weighted spheroidal wave functions, we find that the behavior of the $l$ mode (where $l \geq l_0$) at the asymptotic region of null infinity $scri_+$ is dominated by the following effective Green's function:

$$ G_l^C(z, z'; t) = \sum_{k=l_0}^{l} MF_2 y'^{k+1} v^{-s} {}_sY_l(\theta) {}_sY_k^*(\theta') \times a^{l-k} u^{-(l-s+2)}, \tag{13} $$

where $F_2 = F_2(l, k, m, s) = (-1)^{(l+k+2s+2)/2} 2^k (k + s)!\,(l - s + 1)!\, C_{kl}/\pi(2k + 1)!$.

*Asymptotic behavior at the black-hole outer horizon.*—The asymptotic solution to the homogeneous equation $\tilde{D}(w)\tilde{\Psi}_1(y, w) = 0$ at the black-hole outer horizon $H_+$ ($y \to -\infty$) is $\tilde{\Psi}_1(y, w) = C(w)\Delta^{-s/2} e^{-i(w-mw_+)y}$ [12], where $w_+ = a/2Mr_+$ [$r_+ = M + (M^2 - a^2)^{1/2}$ is the location of the black-hole outer horizon]. In addition, we use $\tilde{\Psi}_1(y', w) \simeq y'^{l+1}$. Regularity of the solution requires $C$ to be an analytic function of $w$. We thus expand $C(w) = C_0 + C_1 w + \ldots$ for small $w$ (as already explained, the late-time behavior of the field is dominated by the low-frequency contribution to the Green's function). Substituting this in Eq. (10), and using the representation Eq. (11) for the spin-weighted spheroidal wave functions, we find that the asymptotic behavior of the $l$ mode (where $l \geq l_0$) at the black-hole outer horizon $H_+$ is dominated by the following effective Green's function:

$$ G_l^C(z, z'; t) = {}_s\Gamma_l M F_1 \Delta^{-s/2} y'^{l_0+1} {}_sY_l(\theta) {}_sY_{l_0}^*(\theta') \times a^{l-l_0} e^{imw_+y} v^{-(l+l_0+3+b)}, \tag{14} $$

where ${}_s\Gamma_l$ are constants, and $b = 0$ generically, except for the unique case $m = 0$ with $s > 0$, in which $b = 1$ [19].

*Pure initial pulse.*—So far we have assumed that the initial pulse consists of all the allowed ($l \geq l_0$) modes. If, on the other hand, the initial angular distribution is characterized by a pure spin-weighted spherical harmonic function ${}_sY_{l^*}^m$, then the asymptotic late-time tails are dominated by modes which, in general, have an angular distribution different from the original one (a full analysis of this case is given in [4]). We find that the field's behavior at the asymptotic regions of timelike infinity $i_+$ and at the black-hole outer horizon $H_+$ is dominated again by the lowest allowed mode (i.e., $l = l_0$). The damping exponents are (in absolute value) $l^* + l_0 + 3 - q$ and $l^* + l_0 + 3 - q + b$, respectively, where $q = \min(l^* - l_0, 2)$.

On the other hand, the behavior of gravitational (and electromagnetic) perturbations at the asymptotic region of null infinity $scri_+$ is dominated by the $l = l_0$ mode if $l_0 \leq l^* \leq l_0 + 2$ and by the $l_0 \leq l \leq l^* - 2$ modes otherwise [4]. The corresponding damping exponents are $l_0 - s + 2$ and $l^* - s$, respectively.

*Summary and physical implications.*—We have analyzed the dynamics of gravitational (physically, the most interesting case) and electromagnetic fields in realistic rotating black-hole spacetimes. The main results and their physical implications are as follows:

(1) We have shown that the late-time evolution of realistic rotating gravitational collapse is characterized by inverse power-law decaying tails at the three asymptotic regions: timelike infinity $i_+$, future null infinity $scri_+$, and the black-hole outer horizon $H_+$ (where the power-law behavior is multiplied by an oscillatory term, caused by the dragging of reference frames at the event horizon). The relaxation of the fields is in accordance with the no-hair conjecture [1]. This Letter reveals the dynamical physical mechanism behind this conjecture in the context of rotating gravitational collapse.

(2) The unique and important feature of rotating gravitational collapse is the active coupling between modes of different $l$ (but the same $m$). Physically, this phenomena is caused by the dragging of reference frames, due to the black-hole (or star's) rotation (this phenomena is absent in the nonrotating $a = 0$ case). As a consequence, the late-time evolution of realistic rotating gravitational collapse has an angular distribution which is generically different from the original angular distribution (in the initial pulse).

(3) The power indices at a fixed radius are found to be $l + l_0 + 3$. These damping exponents are generically smaller than the corresponding power indices in spherically symmetric spacetimes. This implies a slower decay of perturbations in rotating Kerr spacetimes. Stated in a more pictorial way, a rotating Kerr black hole generically becomes "bald" slower than a spherically symmetric Schwarzschild black hole.

(4) It has been widely accepted that the late-time tail of gravitational collapse is universal in the sense that it is independent of the type of the massless field considered (e.g., scalar, neutrino, electromagnetic, and gravitational). This belief was based on spherically symmetric analyses. Our analysis, however, turns over this point of view. In particular, the power indices $l + l_0 + 3$ at a fixed radius which are found in this Letter are generically different from those obtained in the scalar field toy model [9,10] $l + |m| + p + 3$ (where $p = 0$ if $l - |m|$ is even, and $p = 1$ otherwise).

We have shown that different types of fields have different decaying rates. This is a rather surprising conclusion, which has been overlooked in the last three decades. It should be stressed, therefore, that the results obtained from the scalar field toy model [9,10] are actually not applicable for the physically interesting case of higher-spin perturbations (i.e., gravitational and electromagnetic fields).

(5) Our results should have important implications for the mass-inflation scenario and the stability of Cauchy horizons (see, e.g., [20,21] and references therein). In particular, the late-time tails found in this Letter should be used as initial data for perturbations propagating inside the (rotating) black hole.

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