Wigner Symmetry in the Limit of Large Scattering Lengths

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We note that, in the limit where the NN $^1S_0$ and $^3S_1$ scattering lengths, $a^{(^1S_0)}$ and $a^{(^3S_1)}$, go to infinity, the leading terms in the effective field theory for strong NN interactions are invariant under Wigner's SU(4) spin-isospin symmetry. This explains why the leading effects of radiation pions on the S-wave NN scattering amplitudes vanish as $a^{(^1S_0)}$ and $a^{(^3S_1)}$ go to infinity. The implications of Wigner symmetry for $NN \to NN$ axion and $\gamma d \to np$ are also considered.

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Wigner first proposed that SU(4) spin-isospin transformations are an approximate symmetry of the strong interactions [1]. The implications of this symmetry for nuclear physics were studied in Ref. [2]. It has been shown that Wigner symmetry is obtained in the large number of colors limit of quantum chromodynamics (QCD) [3]. In this Letter we show that Wigner's SU(4) is a symmetry of the effective field theory for low momentum nucleon interactions in the limit where the S-wave scattering lengths are infinite, and contact interactions with derivatives are neglected.

Effective field theory methods are applicable to nuclear physics [4,5]. Recently a new power counting has been developed for effective field theory in the two-nucleon sector [6,7]. It is appropriate to the case where the scattering lengths $a^{(^1S_0)}$ and $a^{(^3S_1)}$ in the $^1S_0$ and $^3S_1$ channels are large. As $a^{(^1S_0)}$ and $a^{(^3S_1)}$ go to infinity the couplings for the lowest dimension two-body operators flow to a nontrivial fixed point [5,6]. Higher dimension two-body operators (and, if the pion is not integrated out, pion exchange) are corrections that can be treated perturbatively. Neglecting these corrections the effective field theory is scale invariant when the scattering lengths go to infinity. In this paper we note that in this limit the theory is also invariant under Wigner's SU(4) spin-isospin transformations [1],

$$\delta N = i\alpha_{\mu\nu}\sigma^\mu\tau^\nu N, \qquad N = \begin{pmatrix} p \\ n \end{pmatrix}. \tag{1}$$

In Eq. (1), $\sigma^\mu = (1, \vec{\sigma})$, $\tau^\nu = (1, \vec{\tau})$, and $\alpha_{\mu\nu}$ are infinitesimal group parameters (we will use the notation that Greek indices run over $\{0, 1, 2, 3\}$, while Roman indices run over $\{1, 2, 3\}$). The σ matrices act on the spin degrees of freedom, and the τ matrices act on the isospin degrees of freedom. [Actually the transformations in Eq. (1) correspond to the group SU(4) $\times$ U(1). The additional U(1) is a baryon number and corresponds to the α_{00} term.]

Consider first the effective field theory for nucleon strong interactions with the pion degrees of freedom integrated out. The Lagrange density is composed of nucleon fields and has the form $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \ldots$, where $\mathcal{L}_n$ denotes the n-body terms. We have

$$\begin{aligned} \mathcal{L}_1 &= N^\dagger[i\partial_t + \vec{\nabla}^2/(2M)]N + \ldots, \\ \mathcal{L}_2 &= -\sum_s C_0^{(s)}(N^T P_i^{(s)} N)^\dagger (N^T P_i^{(s)} N) + \ldots, \end{aligned} \tag{2}$$

where M is the nucleon mass and the ellipses denote higher derivative terms. Here $s = {}^1S_0$ or $^3S_1$, and the matrices $P_i^{(s)}$ project onto spin and isospin states

$$P_i^{(^1S_0)} = \frac{(i\sigma_2)(i\tau_2\tau_i)}{\sqrt{8}}, \qquad P_i^{(^3S_1)} = \frac{(i\sigma_2\sigma_i)(i\tau_2)}{\sqrt{8}}. \tag{3}$$

The Lagrange density $\mathcal{L}_2$ can also be written in a different operator basis:

$$\mathcal{L}_2 = -\frac{1}{2}[C_0^S(N^\dagger N)^2 + C_0^T(N^\dagger\vec{\sigma}N)^2] + \ldots, \tag{4}$$

where $C_0^{(^1S_0)} = C_0^S - 3C_0^T$ and $C_0^{(^3S_1)} = C_0^S + C_0^T$. In this basis it is the C_0^T term that breaks the SU(4) symmetry (as well as some of the higher derivative terms).

Neglecting higher dimension operators in Eq. (2), the $^1S_0$ and $^3S_1$ NN scattering amplitudes arise from the sum of bubble Feynman diagrams shown in Fig. 1. The loop integration associated with a bubble has a linear ultraviolet divergence and consequently the values of the coefficients $C_0^{(s)}$ depend on the subtraction scheme adopted. In this paper we use dimensional regularization as the regulator. In minimal subtraction the coefficients are subtraction point independent, and the center of mass scattering amplitude is

$$\mathcal{A}^{(s)} = \frac{-\bar{C}_0^{(s)}}{1 + i\frac{Mp}{4\pi}\bar{C}_0^{(s)}}, \tag{5}$$

where the bar is used to denote minimal subtraction and p is the magnitude of the nucleon momentum. The

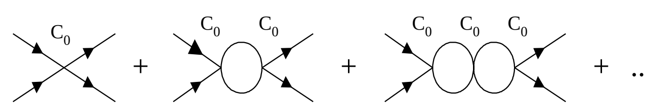

FIG. 1. The leading order contribution to the NN scattering amplitude.

S-wave scattering amplitudes can be expressed in terms of the phase shifts $\delta^{(s)}$,

$$\mathcal{A}^{(s)} = \frac{4\pi}{M} \frac{1}{p \cot\delta^{(s)} - ip}, \tag{6}$$

and it is conventional to expand $p \cot\delta^{(s)}$ in a power series in p^2,

$$p \cot\delta^{(s)} = -\frac{1}{a^{(s)}} + \frac{1}{2} r_0^{(s)} p^2 + \dots, \tag{7}$$

where $a^{(s)}$ is the scattering length and $r_0^{(s)}$ is the effective range. [Strictly speaking, Eq. (6) holds only in the ${}^1S_0$ channel. The ${}^3S_1$ channel is more complicated because of ${}^3S_1$-${}^3D_1$ mixing; however, the mixing is a small effect.] Comparing Eq. (5) with Eqs. (6) and (7), we see that keeping only the lowest dimension two-body terms corresponds to neglecting the effective range and the higher powers of p^2 in Eq. (7),

$$\mathcal{A}^{(s)} = -\frac{4\pi}{M} \frac{1}{1/a^{(s)} + ip}, \tag{8}$$

and that

$$\bar{C}_0^{(s)} = \frac{4\pi a^{(s)}}{M}. \tag{9}$$

If $a^{(s)}$ is of natural size then the dimension six operators in Eq. (2) are irrelevant operators. It is then appropriate to perform a perturbative expansion of the amplitude in a power series in $\bar{C}_0^{(s)}$, which corresponds to an expansion in $pa^{(s)}$. Terms cubic in $\bar{C}_0^{(s)}$ are not more important than the tree level contribution of two-body operators with two derivatives. This situation would be similar to the familiar application of chiral perturbation theory to $\pi\pi$ scattering. However, in nature the scattering lengths are very large: $a^{({}^1S_0)} = -23.714 \pm 0.013$ fm and $a^{({}^3S_1)} = 5.425 \pm 0.001$ fm, or $1/a^{({}^1S_0)} = -8.3$ MeV and $1/a^{({}^3S_1)} = 36$ MeV [8]. The coefficients $\bar{C}_0^{(s)}$ are large and are very different in the ${}^1S_0$ and ${}^3S_1$ channels. Nonetheless, for $p \gg 1/a^{(s)}$, the amplitudes become $\mathcal{A}^{(s)} = 4\pi i/(Mp)$. The equality of the ${}^1S_0$ and ${}^3S_1$ amplitudes is consistent with expectations based on Wigner symmetry. The p dependence is consistent with expectations based on scale invariance, since the cross section $\sigma^{(s)} = 4\pi/p^2$. (The scale transformations appropriate for the nonrelativistic theory are $x \to \lambda x$, $t \to \lambda^2 t$, and $N \to \lambda^{-3/2} N$.)

In minimal subtraction, if $p \gg 1/a^{(s)}$, successive terms in the perturbative series represented by Fig. 1 get larger and larger. Subtraction schemes have been introduced where each diagram in Fig. 1 is of the same order as the sum. It is in these "natural" schemes that the fixed point structure of the theory and Wigner spin-isospin symmetry are manifest in the Lagrangian. One such scheme is power divergence subtraction (PDS) [6], which subtracts not only poles at $D = 4$ but also the poles at $D = 3$ (which correspond to linear divergences). Another such scheme is the off-shell momentum subtraction scheme (OS) [5,9]. In these schemes the coefficients are subtraction point dependent, $C_0^{(s)} \equiv C_0^{(s)}(\mu)$. Calculating the bubble sum in PDS or OS gives

$$\mathcal{A}^{(s)} = -\frac{C_0^{(s)}(\mu)}{1 + \frac{M}{4\pi}(\mu + ip) C_0^{(s)}(\mu)}, \tag{10}$$

where

$$C_0^{(s)}(\mu) = -\frac{4\pi}{M} \frac{1}{\mu - 1/a^{(s)}}. \tag{11}$$

For $\mu \sim p$ the contribution of every diagram in the sum in Fig. 1 is roughly the same size. Furthermore, as $a^{(s)} \to \infty$ the coefficients $C_0^{(s)}(\mu) \to -4\pi/(M\mu)$ which is the same in both channels. In this limit $C_0^T(\mu) = [C_0^{({}^3S_1)}(\mu) - C_0^{({}^1S_0)}(\mu)]/4 = 0$ and

$$\mathcal{L}_2 = -\frac{2\pi}{M\mu} (N^\dagger N)^2 + \dots. \tag{12}$$

The first term in Eq. (12) is invariant under the Wigner spin-isospin transformations in Eq. (1). The ellipses in Eq. (12) denote terms with derivatives, and they will not be invariant under Wigner symmetry even in the limit $a^{(s)} \to \infty$. However, these terms are corrections to the leading order Lagrange density and their effects are suppressed by powers of p/Λ (where Λ is a scale determined by the pion mass and $\Lambda_{\rm QCD}$). In the region $1/a^{(s)} \ll p \ll \Lambda$, Wigner spin-isospin symmetry is a useful approximation and deviations from this symmetry are suppressed by $C_0^T(\mu) \propto (1/a^{({}^1S_0)} - 1/a^{({}^3S_1)})$ and by powers of p/Λ. The measured effective ranges are $r_0^{({}^1S_0)} = 2.73 \pm 0.03$ fm and $r_0^{({}^3S_1)} = 1.749 \pm 0.008$ fm [8]. A rough estimate of the scale is $1/\Lambda \sim [r_0^{({}^1S_0)} - r_0^{({}^3S_1)}]/2 = 0.49$ fm, or $\Lambda \sim 400$ MeV. In PDS or OS, the limit $a^{(s)} \to \infty$ is clearly a fixed point of $C_0^{(s)}(\mu)$ since $\mu\partial/\partial\mu[\mu C_0^{(s)}(\mu)] = 0$. Also, scale invariance is manifest since $\mu \to \mu/\lambda$ under scale transformations.

Wigner symmetry is useful even though $a^{({}^1S_0)}$ and $a^{({}^3S_1)}$ are very different. This is because, for $1/a^{(s)} \ll p \ll \Lambda$, corrections to the symmetry limit go as $(1/a^{({}^1S_0)} - 1/a^{({}^3S_1)})$ rather than $(a^{({}^1S_0)} - a^{({}^3S_1)})$. This is similar to the heavy quark spin-flavor symmetry of QCD [10], which occurs in the $m_Q \to \infty$ limit. Heavy quark symmetry is a useful approximation for charm and bottom quarks even though $m_b/m_c \simeq 3$.

As an application of the symmetry, consider $NN \to NN$ axion, which is relevant for astrophysical bounds on the axion coupling [11]. The axion is essentially massless. If the axion has momentum $\vec{k}$, and the initial nucleons have momenta $\vec{p}$ and $-\vec{p}$ then the final state nucleons have momenta $\vec{q} - \vec{k}/2$ and $-\vec{q} - \vec{k}/2$. Energy conservation implies that $p^2/M = q^2/M + k^2/(4M) + k$, where $p = |\vec{p}|$, $q = |\vec{q}|$, and $k = |\vec{k}|$. In the kinematic region we consider $q, p \gg k$, and the axion momentum can be neglected in comparison with the nucleon momenta. In this limit the terms in the Lagrange density which couple the axion to nucleons take the form

$$\mathcal{L}_{\rm int} = g_0 (\nabla^j X^0)|_{\vec{x}=0} N^\dagger \sigma^j N + g_1 (\nabla^j X^0)|_{\vec{x}=0} N^\dagger \sigma^j \tau^3 N , \tag{13}$$

where X^0 is the axion field and g_0, g_1 are the axion-nucleon isosinglet and isovector coupling constants. Associated with spin-isospin symmetry are the conserved charges

$$Q^{\mu\nu} = \int d^3 x N^\dagger \sigma^\mu \tau^\nu N, \tag{14}$$

and the axion terms in the action are proportional to these charges

$$S_{\rm int} = g_0 \int dt (\nabla^j X^0) \Big|_{\vec{x}=0} Q^{j0} + g_1 \int dt (\nabla^j X^0) \Big|_{\vec{x}=0} Q^{j3}. \tag{15}$$

The charge Q^{j0} is the total spin of the nucleons which is conserved even without taking the $a^{(s)} \to \infty$ limit; however, Q^{j3} is conserved only in the $a^{(s)} \to \infty$ limit (and also in the limit $a^{(^1S_0)} \to a^{(^3S_1)}$). Since conserved charges are time independent, only a zero energy axion couples in Eq. (15), and these terms will not contribute to the scattering amplitude. We conclude that $NN(^1S_0) \to NN(^3S_1)X^0$ vanishes in the limit $a^{(s)} \to \infty$ and that $NN(^3S_1) \to NN(^3S_1)X^0$ vanishes for all scattering lengths. [$NN(^1S_0) \to NN(^1S_0)X^0$ vanishes due to angular momentum conservation since the axion is emitted in a P wave.] Calculation of the Feynman diagrams in Fig. 2 shows that the leading order $^3S_1 \to {}^3S_1$ scattering amplitude does indeed vanish, and the $NN(^1S_0) \to NN(^3S_1)X^0$ amplitude is

$$\mathcal{A} = g_1 \frac{4\pi}{M} \frac{\vec{k} \cdot \vec{\epsilon}^*}{k} \left[\frac{1}{a^{(^1S_0)}} - \frac{1}{a^{(^3S_1)}} \right] \left[\frac{1}{1/a^{(^1S_0)} + ip} \right] \times \left[\frac{1}{1/a^{(^3S_1)} + iq} \right], \tag{16}$$

where $\vec{\epsilon}$ is the polarization of the final $^3S_1$ NN state. This is proportional to $(1/a^{(^1S_0)} - 1/a^{(^3S_1)})$ and is consistent with our expectations based on the Wigner symmetry.

Coupling of photons to nucleons occurs by gauging the strong effective field theory and by adding terms involving the field strengths $\vec{E}$ and $\vec{B}$. In the kinematic regime where the photon's momentum is small compared to the nucleons' momentum, the part of the action involving the field strengths is

$$S_{\rm int} = \frac{e}{2M} \int dt\, B^j \Big|_{\vec{x}=0} (\kappa_0 Q^{j0} + \kappa_1 Q^{j3}) + \ldots, \tag{17}$$

where κ_0 and κ_1 are the isosinglet and isovector nucleon magnetic moments in nuclear magnetons, and the ellipses denote subdominant terms. The term proportional to κ_1 in Eq. (17) gives the lowest order contribution to the amplitude for $\gamma d \to np(^1S_0)$. The form of the coupling above implies that, like the axion case, this amplitude is proportional to $(1/a^{(^1S_0)} - 1/a^{(^3S_1)})$.

So far we have considered an effective field theory with the pions integrated out. It is straightforward to include the pion fields, and this is expected to increase the range of validity of the momentum expansion. Pion exchange can be separated into two types, potential and radiation. Potential pions have $k^0 \sim k^2/M$, where k^0 is the pion energy and k is the magnitude of the pion momentum. Radiation pions are nearly on-shell; i.e., $k^0 \sim \sqrt{k^2 + m_\pi^2}$. With the power counting in Ref. [6], $C_0^{(s)}(\mu)$ gives the leading order S-wave NN scattering amplitude, while potential pion exchange and four-nucleon operators with two derivatives enter at next-to-leading order. As our last example, we discuss the corrections to NN scattering due to radiation pions [12]. As pointed out in Ref. [12], one should perform a multipole expansion on the coupling of radiation pions to nucleons. The first term in the multipole expansion is

$$S_{\rm int} = -\frac{g_A}{\sqrt{2} f} \int dt (\nabla^i \pi^j) \Big|_{\vec{x}=0} Q^{ij} , \tag{18}$$

where $g_A \simeq 1.25$ is the axial coupling and $f \simeq 131$ MeV is the pion decay constant. (Radiation gluons in NRQCD and radiation photons in NRQED are also treated in this way [13].) Radiation pions also couple to a conserved charge of the Wigner symmetry in the large scattering length limit. (A multipole expansion is not performed on the coupling to potential pions so they do not couple to

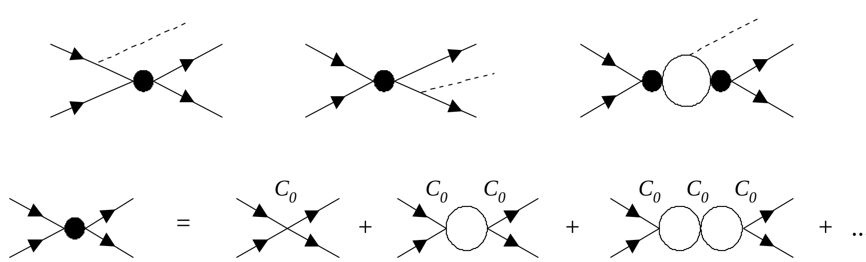

FIG. 2. Graphs contributing to $NN \to NN$ axion at leading order. The solid lines denote nucleons and the dashed lines are axions.

a conserved charge.) This implies that only a radiation pion with $k^0 = 0$ will couple, which is incompatible with the condition $k^0 \sim \sqrt{k^2 + m_\pi^2}$, so in the symmetry limit radiation pions do not contribute to the scattering matrix element. In Ref. [12], it was shown by explicit computation that graphs with one radiation pion and any number of $C_0^{(s)}$'s give a contribution that is suppressed by at least one power of $1/a^{({}^3S_1)} - 1/a^{({}^1S_0)}$. This suppression was the result of cancellations between many different Feynman diagrams. Wigner symmetry guarantees that the leading contribution of graphs with an arbitrary number of radiation pions is suppressed by inverse powers of the scattering lengths.

So far, the analysis in this paper has been specific to the two-nucleon sector; however, Wigner symmetry is observed in some nuclei with many nucleons. Terms with no derivatives also occur in $\mathcal{L}_3$ and $\mathcal{L}_4$, while higher-body terms of this type vanish because of Fermi statistics. Fermi statistics implies that there is only one no-derivative four-body term, $(N^\dagger N)^4$, which is invariant under Wigner symmetry. Furthermore, there is only one such term in $\mathcal{L}_3$, $(N^\dagger N)^3$, which is also invariant [14]. To see this, note that the three nucleon and antinucleon fields must be combined in an antisymmetric way. The three N's ($N^\dagger$'s) combine to a $\bar{4}$ (4) of SU(4). Combining the 4 and $\bar{4}$ gives $1 \oplus 15$; however, the 15 does not contain a singlet under the spin and isospin SU(2) subgroups.

Recent progress [15] in the three-body sector suggests that the $(N^\dagger N)^3$ contact interaction is not subleading compared with the effects of the first two-body term in Eq. (12). (For another point of view see Ref. [16].) If the higher-body operators with derivatives can be treated as perturbations, then this Letter suggests that approximate Wigner symmetry in nuclear physics is a consequence of the large NN scattering lengths and some simple group theory.

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