Origin of the "−1" Spectral Law in Wall-Bounded Turbulence

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The existence of the "−1" spectral law in wall turbulence is explained by the effect of superposition of Kolmogorov's eddy cascades generated at all possible distances from the wall, within an equilibrium layer. This concept is justified using only the well-known properties of wall-bounded flows. The presence of coherent structures appears to be not essential for the inverse-power law. The model predicts the -1 scaling for the range $(1/H) \leq k \leq (1/z)$, which agrees very well with the measurements (k is the wave number, H is the external scale of the flow, and z is the distance from the wall).

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According to the original Kolmogorov's [1] concept, velocity spectra consist of three ranges: (1) the production range, where spectral behavior has not been identified specifically; (2) the inertial subrange, where spectra follow the "$-5/3$" law (there is no energy production or dissipation in this subrange); and (3) the viscous range, where spectra decay much faster than in the inertial subrange, due to dissipation. In 1953 Tchen [2] supplemented this model, for shear flows without solid boundaries, with one more region intermediate between regions (1) and (2). In this region, the wave-number spectrum $S_{ii}(k)$ decays as $S_{ii}(k) \sim k^{-1}$, where k is the longitudinal wave number in the direction of the mean flow where homogeneity applies. From Tchen's [2] analysis it follows that the -1 and $-5/3$ scaling regions are the result of strong (-1) and weak ($-5/3$) interactions between the mean flow and fluctuation vorticities. He also suggested that the -1 region should exist in velocity cospectra $S_{ij}(k)$ [note that in the inertial subrange $S_{ij}(k) \sim k^{-7/3}$].

After Tchen's [2] pioneering work, several new explanations of the -1 spectral scaling have appeared. They fall into three groups: (1) further developments of Tchen's spectral balance approach [3,4]; (2) similarity and dimensional analysis [5,6]; and (3) analysis of near-wall coherent structures ([7], and references therein). Although numerous measurements (see, e.g., Ref. [4] for review) confirm the -1 scaling, its origin is still unclear. In this paper we propose an alternative explanation for the origin of the -1 scaling which does not require consideration of coherent structures and which is based on simple phenomenology developed for wall turbulence.

In our considerations we use the following properties of wall turbulence (close to the wall, within the logarithmic layer): (A) The shear stress τ is approximately constant and equal to $\tau = \rho u_*^2$ (u_* is the friction velocity, and ρ is fluid density). The production of the total turbulence energy P is approximately equal to the energy dissipation ε_d that leads to $P \approx \varepsilon_d \sim u_*^3/z$ (Fig. 1). These properties describe Townsend's [8] equilibrium wall layer with constant shear stress. (B) The mean flow instability and velocity shear generate a hierarchy of eddies attached (in the sense of Townsend [8]) to the wall so that their characteristic scales are proportional to the distance z from the wall (Fig. 1).

The above two properties are well tested and accepted in wall turbulence studies. Using property B we can reasonably assume that due to flow instability and velocity shear the energy injection from the mean flow into turbulence occurs at each distance z from the wall, with the generation of eddies with characteristic scale $L \sim z$. These eddies transfer their energy at rate ε to smaller eddies and may be viewed as energy cascade initiators. In other words, we suggest that at each z a separate Kolmogorov's cascade is initiated which is superposed with other eddy cascades initiated at other z's. Figure 2 presents a sketch explaining this process. As a result of this superposition of cascades, the energy dissipation ε_d at a particular distance z presents a superposition of down-scale energy fluxes, ε, generated at this and at larger z (the contribution from cascades generated at smaller z is negligible; justification for this may be

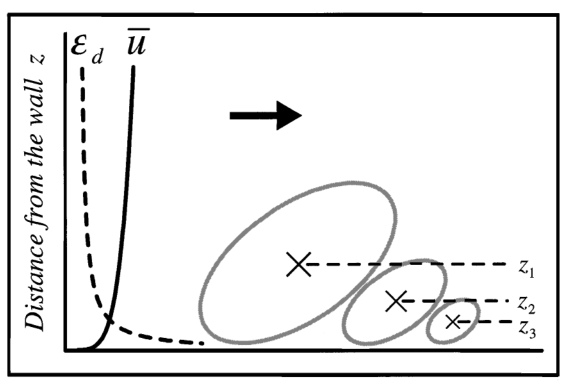

FIG. 1. A wall-bounded flow structure showing the vertical distributions of local mean velocity $\overline{u}$ and turbulence energy dissipation ε_d. Ellipses represent eddies generated at distances z_1, z_2, and z_3 (arbitrarily selected for clarity) as a result of mean flow instability. Note that exact eddy morphology is not essential for this consideration.

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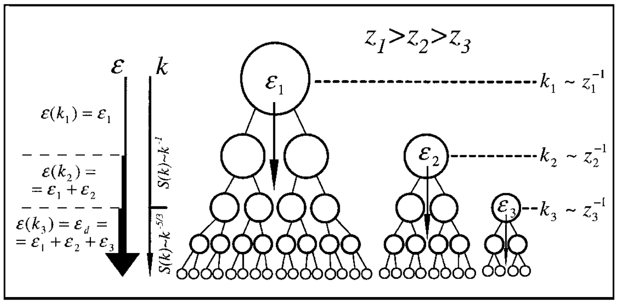

FIG. 2. The effect of superposition of energy cascades measured at z_3 and initiated at all possible z. Note that for clarity only three cascades initiated at distances z_1, z_2, and z_3 are shown, as in Fig. 1.

found in Townsend [8]). Thus, the energy flux ε at any z depends on the scale under consideration, i.e., on wave number k. The flux ε increases with k until it reaches $1/z$ and then, for $k \geq (1/z)$, stabilizes being equal to ε_d (Figs. 2 and 3). In other words, at a given distance z_g the energy flux $\varepsilon(k)$ for $k < z_g^{-1}$ represents the energy dissipation ε_d observed at $z = k^{-1}$; $z > z_g$. Using property A (i.e., $\varepsilon \sim u_*^3/z$) and bearing in mind that $L \sim z \sim k^{-1}$ as in Fig. 2, we have $\varepsilon(k) \sim u_*^3 k$ for $(1/H) \leq k \leq (1/z)$. The scale H is an external scale of the flow (e.g., thickness of a boundary layer). Following this phenomenological concept and using the inertial subrange relationship $S_{ii}(k) \sim \varepsilon_d^{2/3} k^{-5/3}$ we can write for the autospectra

$$S_{ii}(k) \sim \varepsilon(k)^{2/3} k^{-5/3} \sim u_*^2 k^{-1} \quad \text{for } (1/H) \leq k \leq (1/z)\,, \tag{1}$$

and

$$S_{ii}(k) \sim \varepsilon(k)^{2/3} k^{-5/3} \sim \varepsilon_d^{2/3} k^{-5/3} \quad \text{for } k \geq (1/z)\,. \tag{2}$$

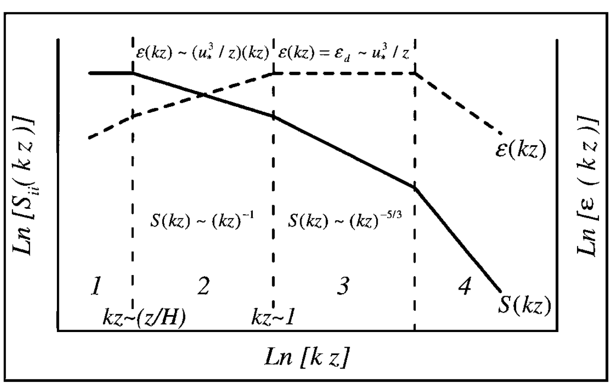

FIG. 3. Schematized velocity autospectra $S_{ii}(kz)$ and energy transfer rate $\varepsilon(kz)$ showing (1) the large-scale energy production range ($k > H^{-1}$); (2) the -1 scaling range ($H^{-1} < k < z^{-1}$), where eddy cascades initiated at each z are superposed and $\varepsilon(k)$ changes as $\varepsilon(k) \sim k$; (3) the inertial subrange ($k > z^{-1}$) which results from superposition of inertial subranges generated at each z and, therefore, $\varepsilon(k) \equiv \varepsilon_d$; and (4) the dissipative range.

Accounting for Eqs. (1) and (2), velocity autospectra for wall-bounded flows can be summarized as in Fig. 3. The inverse Fourier transform of Eqs. (1) and (2) produces the second order structure function $D(r)$ that shows Kolmogorov's "2/3" scaling at $r < z$ $[D(r) \sim r^{2/3}]$ and quasilogarithimic behavior at $z \ll r \ll H$ $[D(r) \sim \ln(r/z) + f(r/H)]$.

Similar considerations are also valid for velocity cospectra. According to Tchen [2], Lumley [9], and Wyngaard and Cote [10], the cospectrum $S_{uw}(k)$ in the inertial subrange is $S_{uw}(k) \sim \varepsilon_d^{1/3}(d\overline{u}/dz)k^{-7/3}$ (u and w are the longitudinal and vertical velocities, respectively; $\overline{u}$ is the local mean velocity). Following our concept and using property A, i.e., $d\overline{u}/dz \sim u_*/z$ and $\varepsilon \sim u_*^3/z$, this relationship becomes $S_{uw}(k) \sim u_*^{-2}\varepsilon_d^{4/3}k^{-7/3}$; thus

$$S_{uw}(k) \sim u_*^{-2}\varepsilon(k)^{4/3}k^{-7/3} \sim u_*^2 k^{-1} \quad \text{for } (1/H) \leq k \leq (1/z)\,, \tag{3}$$

and

$$S_{uw}(k) \sim u_*^{-2}\varepsilon(k)^{4/3}k^{-7/3} \sim u_*^{-2}\varepsilon_d^{4/3}k^{-7/3} \quad \text{for } k \geq (1/z)\,. \tag{4}$$

The above considerations are based on the concept of the nonintermittent energy cascade known as $K41$ [1,11,12]. This concept has been found to be in good agreement with measurements for moments of the order of 2 or less [11,12]. Therefore, in this note we restrict ourselves to this simple concept, though intermittency corrections (which are negligible for the second order moments) can be also incorporated.

The -1 scaling has been originally predicted [2–4,13] for regions in shear flows, not necessarily near the wall, where the ratio $m = (dU/dz)/(\varepsilon/\nu)^{0.5}$ of the mean shear dU/dz to the turbulent shear $(\varepsilon/\nu)^{0.5}$ is not small, i.e., of the order of 1. However, published data reveal the -1 spectral scaling in the near-wall flow regions only, independently of $(dU/dz)(\nu/\varepsilon)^{0.5}$ (see [4] for a recent review). For example, velocity spectra of Champagne *et al.* [14] in the nearly homogeneous turbulent shear flow at $m \approx 0.1$ do not show the -1 scaling while near-wall measurements of Antonia and Raupach [15] reveal such a scaling even at smaller $m \approx 0.06$. Such behavior contradicts [2–4,13] but is consistent with the model presented in this Letter. This model may also be extended to the equilibrium wall layers with variable stress [8]. All these suggest that the -1 spectral scaling is an exclusive feature of Townsend's [8] equilibrium wall layers and should not exist in homogeneous shear or free shear flows.

In conclusion, this Letter proposes a simple phenomenological model explaining the -1 law in wall turbulence as a result of superposition of eddy cascades generated at all possible z. Only well-known properties of wall-bounded flows have been used. The presence of coherent structures does not appear to be essential for the inverse-power law. The model predicts the -1 scaling for the range $(1/H) \leq k \leq (1/z)$, which agrees very well with measurements.

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