Probability Distribution of the Conductance at the Mobility Edge

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(Received 26 January 1999)

The distribution of the conductance $P(g)$ at the critical point of the metal-insulator transition is presented for three- and four-dimensional orthogonal systems. The distribution is system-size independent but it depends on the dimension of the system. Its form is discussed and quantitative formulas for limiting cases $g \to 0$ and $g \to \infty$ are derived. The relation $P(g) \to 0$ in the limit $g \to 0$ is proven.

PACS numbers: 71.30.+h, 71.23.–k, 72.15.Rn

As the conductance g in disordered systems is not the self-averaged quantity, the knowledge of its probability distribution is extremely important for our understanding of transport. This problem is of special importance at the critical point of the metal-insulator transition [1]. While the distribution of the conductance in the metallic phase is known to be Gaussian in agreement with the random-matrix theory [2] and the localized regime is characterized by the log-normal distribution of g [2], our knowledge about the critical distribution remains still insufficient. Numerical studies proved the system-size invariance of $P(g)$ at the critical point [3–8], which is consistent with the scaling theory of localization. The shape of the distribution is, however, not completely understood.

Several attempts have been made to characterize conductance distribution at the critical point. Using the Migdal-Kadanoff renormalization treatment, huge conductance fluctuations have been predicted in [3]. The same conclusion was found also in systems of dimension $d = 2 + \varepsilon$. In the limit $\varepsilon \ll 1$ the form of the distribution $P(g)$ was found analytically [4]. However, numerical studies of the disordered 3D orthogonal system [5] indicated that it is not possible to generalize these analytical conclusions for realistic 3D systems ($\varepsilon = 1$). Here, no huge fluctuations of the conductance have been found neither in linear nor in the logarithmic scale [5,8,9,10].

The form of $P(g)$ for the 2D symplectic models was found numerically in [6,7]. Recently, $P(g)$ has been studied also for a system in the magnetic field, both in 3D [8] and in 2D [11]. The main conclusion of these studies is that the symmetry of the system influences the form of the distribution at the critical point more strongly than in the metallic or localized regime.

Besides the symmetry of the system, there are other parameters which could influence the form of $P(g)$. The invariance of $P(g)$ along the critical line in the (energy, disorder) plane in the 3D Anderson model has been proven numerically in [7]. In the same paper it has been shown that two different microscopic 2D symplectic models have the same critical conductance distribution.

The proof of the universality of $P(g)$ within a given universality class would require an analysis of how $P(g)$ scales with the system size in the neighbor of the critical point. It is probably impossible to realize such an analysis numerically. Nevertheless, numerical results for the critical point [7,12] indicate that $P(g)$ is a unique function of the mean value $\langle g\rangle$. This would support the hypothesis of universal scaling of $P(g)$ in spite of the fact that the form of $P(g)$ itself depends for a given model on some microscopical parameters (e.g., anisotropy) [12] and even on the choice of the boundary conditions [10].

Studies of the statistics of the conductance have their counterpart in the analysis of the level statistics $s = E_{i+1} - E_i$ of the eigenvalues of the Hamiltonian [13]. The critical distribution $P(s)$ is also the subject of intensive studies within the past years [14]. In particular, its dependence on the symmetry [15,16], dimension [17], and boundary condition [18] has been studied numerically.

In this Letter we present new numerical data for the conductance distribution $P(g)$ at the critical point of the 3D and 4D Anderson models (orthogonal ensembles). Our data confirm the system-size invariance of $P(g)$ and prove that the critical distribution depends on the dimension. Then we present also a quantitative analysis of the form of $P(g)$ in limits $g \to 0$ and $g \to \infty$. In particular, we prove that $P(g) \to 0$ for $g \to 0$.

Numerical analysis of the form of $P(g)$ started in [5], and later in [8]. Recently, small-g behavior was studied in detail in [9,10]. Our results confirm quantitative behavior of $P(g)$ as has been found in [5,10].

We calculated the conductance as

$$g = \mathrm{Tr}\, t^\dagger t = \sum \cosh^{-2}(z_i/2)\,, \qquad (1)$$

where quantities z_i determine eigenvalues of the transmission matrix $t^\dagger t$ for the disordered system $L^{d-1} \times L$ (with periodic boundary condition). Details of the method have been published elsewhere [5]. For a given system size L, the probability distribution of g has been calculated from an ensemble of $N_{\rm stat}$ samples. The list of used ensembles together with mean and variances of g is given in Table I.

The last column of Table I presents parameter $\langle z_1\rangle$, which corresponds to the parameter Λ introduced in the finite-size scaling theory by MacKinnon and Kramer [19]

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TABLE I. Review of ensembles studied in the present work. L: size of the d-dimensional cube; N_{stat}: number of samples in a given ensemble; var$g = \langle g\rangle^2 - \langle g^2\rangle$; $\langle z_1\rangle$ is mean of the smallest of z's. Data for the 3D Anderson model are in good agreement with [11] (up to the spin degeneracy factor 2).

L	Symbol	N_{stat}	$\langle g\rangle$	$\sqrt{\text{var}g}$	$\langle \log g\rangle$	var log$g$	$\langle z_1\rangle$
			3D Anderson model: $W_c \approx 16.5$				
6	○	20.000	0.375	0.324	−1.481	1.344	2.901
8	◁	20.000	0.400	0.333	−1.384	1.251	2.803
10	▷	10.000	0.410	0.337	−1.347	1.229	2.770
12	◇	5.000	0.421	0.340	−1.302	1.199	2.724
14	△	2.500	0.416	0.338	−1.306	1.122	2.725
18	▽	500	0.418	0.329	−1.279	1.083	2.717
			4D Anderson model: $W_c \approx 34.5$ [14]				
4		22.000	0.190	0.247	−2.569	2.301	4.130
5	▽	30.000	0.229	0.270	−2.275	2.006	3.838
6	□	15.000	0.225	0.269	−2.291	2.054	3.852
7	△	7.000	0.239	0.275	−2.193	1.971	3.748
8		200	0.227	0.274	−2.188	1.692	3.790

as $\langle z_1\rangle = \frac{2L_t}{L\Lambda}$ in the quasi-one-dimensional limit $L^{d-1} \times L_t$, $L_t \gg L$. When neglecting the smallest system size, our data confirm the L invariance of $\langle z_1\rangle$ as well as of $\langle g\rangle$ and $\langle \log g\rangle$ and their standard deviations. Owing to higher critical disorder, $\langle z_1\rangle$ is larger in 4D than in 3D. This guarantees that the finite-size effects disappear more quickly in 4D. Therefore, in spite of the fact that computer facilities limited the system size to $L \le 8$ for $d = 4$, obtained data provide us with the relevant information about all parameters of interest.

We presented in Table I both mean values of g and $\log g$ to underline the common features of 3D and 4D distribution: the variance of $\log g$ is of the order of its mean value. This relation is typical for the localized state. On the other hand, the standard deviation of g is also $\sim\langle g\rangle$. Its value for 3D samples, 0.334, is smaller than the same quantity calculated for 3D in the metallic regime [5].

Numerical data for $P(g)$ are presented in Fig. 1. To compare the distributions for different systems, we normalize conductance to its mean value. Our data confirm that the critical distribution of g is system-size independent, in agreement with previous studies. Figure 1 shows also that $P(g)$ depends on the dimension of the system within the same symmetry class. Although the distribution has the same form for 3D and 4D ensembles, it becomes broader for higher d: the probability to find $g \ll \langle g\rangle$ or $g \gg \langle g\rangle$ growths with dimension. This is due to higher critical disorder, which causes the electronic state to possess more features of the localized state than that of the metallic one (remaining critical). This is in agreement with studies of the level statistics in 4D [14].

The small-g behavior of $P(g)$ can be estimated from Fig 1. Instead of $P(g/\langle g\rangle)$, we plot in the left side of Fig. 1 the distribution $\mathcal{P}(\gamma)$ of $\gamma = \log g/\langle g\rangle$. Evidently, $\log\mathcal{P}(\gamma) = \gamma + \log P(\exp\gamma)$. Therefore, an assumption $P(g = 0) = c \neq 0$, implies $\mathcal{P} = \gamma + \log c$ for $\gamma \to -\infty$.

Figure 1 shows clearly that $\log\mathcal{P}(\gamma)$ decreases more quickly than γ for *all* ensembles we consider. This guarantees that $P(g) \to 0$ as $g \to 0$. Let us note that it is almost impossible to obtain the last result from the studies of $P(g)$ on the linear scale [9].

The small-g behavior of $P(g)$ is easy to estimate also from the distribution $P(z_1)$ of the smallest parameter z_1. Indeed, small values of g require large values of z_1.

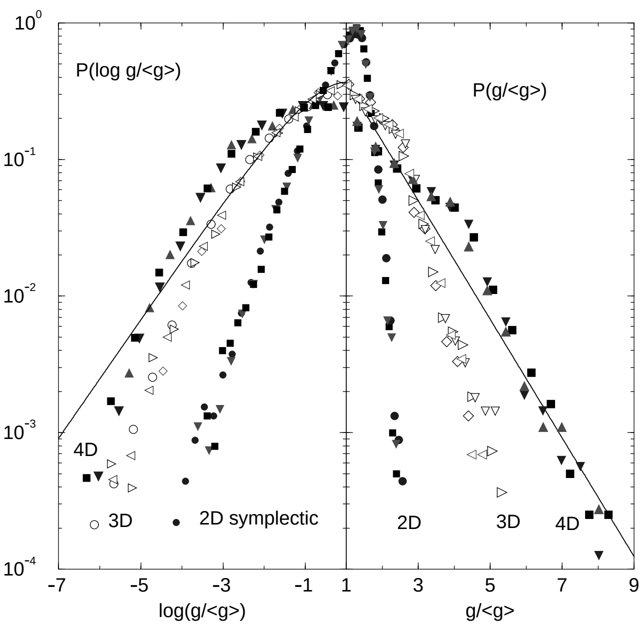

FIG. 1. Probability distribution of $\log g/\langle g\rangle$ (left) and $g/\langle g\rangle$ (right) for 4D (solid symbols) and 3D (open symbols) Anderson model. For comparison, we plot also data for the 2D (symplectic) Ando model. The last model exhibits the best convergence for both small and large values of g. For the meaning of the symbols, see Table I. The solid line is the Poisson distribution $P(g) = \exp(-g/\langle g\rangle)$.

Neglecting contributions of other channels, we have

$$\frac{1}{2\varepsilon}\int_0^{2\varepsilon} P(g)\,dg = \frac{1}{2\varepsilon}\int_{\tilde{z}_1}^{\infty} P(z_1)\,dz_1\,, \qquad (2)$$

with $\varepsilon = \exp(-\tilde{z}_1)$. In the limit $\varepsilon \to 0$ the integral on the left-hand side (LHS) reads $\sim P(g)$, $g = \varepsilon$. The right-hand side (RHS) could be found analytically for the special form of $P(z_1)$. In particular, for Wigner surmises $P(z_1) = \pi/2\langle z_1\rangle^2 \times z_1 \exp(-\pi/4 \times [z_1/\langle z_1\rangle]^2)$ we obtain that $P(g) \sim g^{-1-\mathrm{const}\times\log g}/2$ with const $= \frac{\pi}{4\langle z_1\rangle^2}$. Consequently, $P(g=0) = 0$. Figure 2 assures that $P(z_1)$ decreases more quickly than Wigner surmise for large z_1 in orthogonal ensemble for both 3D and 4D systems. This assures that $P(g) \to 0$ as $g \to 0$.

Linear behavior of the distribution $P(z_1)$ for small z_1 (see left side of Fig. 2) guarantees nonzero probability that the first channel is fully open. Indeed, if $P(z_1) \sim C \times z_1$ for $z_1 \to 0$, then the probability that the first channel contribution to the conductance, $g_1 = 1/\cosh^2(z_1)$, equals to 1, is C. This explains the origin of the characteristic bump in the distribution $P(g)$ for $g = 1$. In Fig. 1, the bump is clearly visible for both 3D and 4D systems.

Figure 1 (RHS) confirms that $P(g)$ decreases more quickly than exponentially for large g. This is easy to understand on the basis of the analysis of the statistics of z's presented in [5].

Figure 3 shows mean values and variances of some smallest z's for both 3D and 4D systems. Evidently, $\langle z_i\rangle \sim \mathcal{O}(1)$ and variance varz_i decreases quickly with index i. Consequently, the contribution to the conductance from the second (higher) channel is, due to (1), small (negligible). To estimate this contribution, we note that all higher z_i, $i \geq 2$, are normally distributed [5]. Their mean and variances have been estimated as $\langle z_i\rangle \sim \langle z_1\rangle \times i^{1/(d-1)}$ and var$z_i \sim \langle z_i\rangle^{-(d-2)}$ [20]. Although this result holds only in the quasi-one-dimensional limit, where the mutual correlations of z's are negligible, they serve as a good quantitative estimation also for true d-dimensional cubes. As i is seen in Fig. 3, this agreement is better for 4D than for 3D. Then, the probability to find $g \approx n$ is less than $\exp[-\langle z_n\rangle/2\mathrm{var}z_n] \sim \exp[-\mathrm{const} \times n^{d/(d-1)}]$ and

$$P(g) \sim \exp - \text{ const} \times g^{d/(d-1)}, \qquad g \to \infty. \qquad (3)$$

We conclude that presented numerical data for 3D and 4D Anderson models prove the system-size invariance of the conductance distribution at the critical point. Although the distribution depends on the dimension and symmetry of the system, we found its common features, namely exponential decrease of $P(g)$ for $g > 1$, and a decrease of $P(g)$ to zero for $g = 0$.

We explained the form of $P(g)$ using the statistical properties of parameters z introduced by relation (1). As the last can be easily analyzed numerically, this treatment opens new possibilities for detailed quantitative description of $P(g)$. Such analysis is more simple for higher dimension, where the statistical correlations of z's are supposed to be less important.

The statistical properties of z's explain also the dimension dependence of $P(g)$ and gives at least a qualitative explanation why the analytical results, obtained in [4] for dimension $d = 2 + \varepsilon$, could not be applied to 3D systems [5]. Our results are consistent with studies of the level statistics in 4D systems [17].

Presented data could not prove the invariance of $P(g)$ with respect to the change of the microscopic parameters

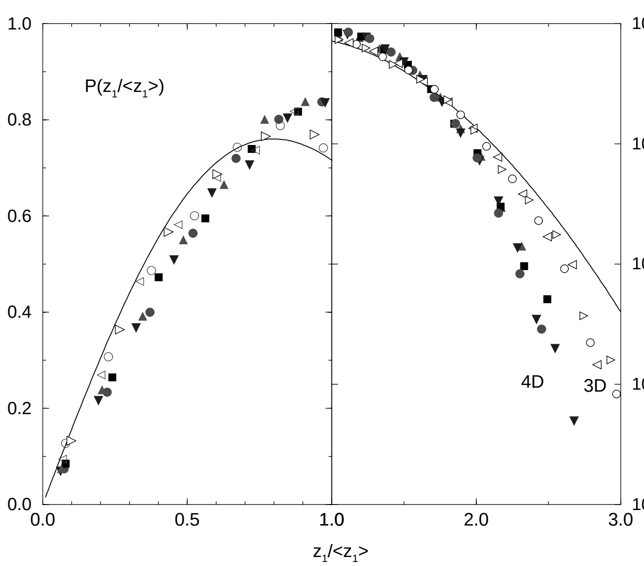

FIG. 2. Probability distribution of (normalized) z_1 for 4D and 3D Anderson models. The solid line is Wigner surmises $P_W(z) = \frac{\pi}{2} z \exp[-\frac{\pi}{4} z^2]$. For the mean value $\langle z_1\rangle$, see Table I.

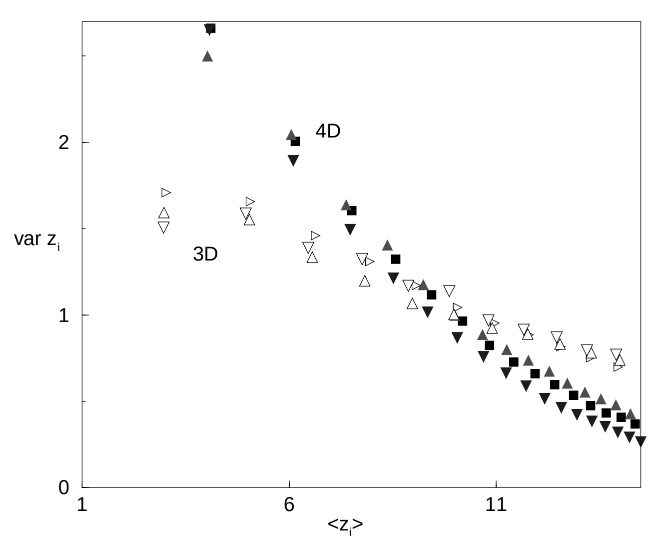

FIG. 3. varz_i as a function of $\langle z_i\rangle$ (only for $\langle z\rangle < 15$) for 3D (open symbols) and 4D (solid symbols) orthogonal systems. Note the system-size invariance of presented parameters (at least for $i \leq L$).

of the model (energy, anisotropy, etc.). However, the common scaling of parameters z in the quasi-one-dimensional limit, found in [21], together with formula (1) indicates that the distribution $P(g)$ is completely determined by only one parameter within the given symmetry class: we can choose it as $\langle z_1\rangle$, $\Lambda_c = 2/\langle z_1\rangle$, or $\langle g\rangle$. This means that two models α and β, with different microscopical parameters (energy, anisotropy, etc.), have the same critical distribution $P(g)$ iff $\Lambda_c(\alpha) = \Lambda_c(\beta)$. Our numerical data for 3D and 2D systems [7] as well as for anisotropic systems [12] confirm this assumption.

This work has been supported by the Slovak Grant Agency, Grant No. 2/4109/98.

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