Experimental Determination of a Nonlinear Dynamic Model of Plasma Turbulence Using Feedback Control

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A new and convenient method to experimentally determine nonlinear dynamical models of plasma turbulence is described. It is based on a novel use of feedback control which generates information about the model without introducing additional unknowns. The method is applied to an $\mathbf{E} \times \mathbf{B}$ turbulence in the Columbia Linear Machine to determine the linear eigenfrequencies and nonlinear coupling coefficients of a three wave coupling model directly from experimental data.

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The physics of turbulent transport in plasmas remains largely an open question. An essential ingredient of this physics issue is an experimentally validated nonlinear dynamical model of plasma turbulence. Direct experimental determination of the nonlinear dynamics underlying the turbulence is nearly an intractable proposition. Experimental verification of a number of well-known theoretical dynamical models is largely absent. In this Letter we describe a novel experimental method for the determination of the parameters of a three wave coupling model which is an often used theoretical tool. First, it should be pointed out that the three wave coupling model is widely used in many areas of physics, i.e., hydrodynamics, nonlinear optics (three wave mixing, parametric superfluorescence), etc. In experimental settings with significant noise, the method developed in this paper is extrapolatable and may be uniquely suitable. Second, the method described below can be easily adapted to another model of nonlinear dynamics of plasma turbulence: nonlinear Landau damping. The method is based on Ritz, Powers, and Bengston [1], which assumes a nonlinear three wave coupling equation in the form of

$$\frac{\partial \phi(k,t)}{\partial t} = (\gamma_k + i\omega_k)\phi(k,t) + \frac{1}{2}\sum_{k=k_1+k_2} \Lambda_k^Q(k_1,k_2)\phi(k_1,t)\phi(k_2,t)\,, \tag{1}$$

where the spatial Fourier spectrum $\phi(k,t)$ of the fluctuating field is defined by $\phi(x,t) = \sum_k |\phi(k,t)| e^{i[\theta(k,t)+kx]}$ [slow varying amplitude with respect to changes in phase $\theta(k,t)$], γ_k denotes the growth rate, ω_k the real frequency and $\Lambda_k^Q(k_1,k_2)$ the coupling coefficient of this process. After discretization and transforming to the frequency domain, Eq. (1) is multiplied by Φ_ω^* and $\Phi_{\omega_1'}^*\Phi_{\omega_2'}^*$, respectively, and ensemble averaged yielding

$$\langle \Phi_\omega(\tau)\Phi_\omega^* \rangle = L_\omega \langle \Phi_\omega \Phi_\omega^* \rangle + \frac{1}{2}\sum_{\omega=\omega_2-\omega_1} Q_{\omega_1,\omega_2}\langle \Phi_{\omega_1}\Phi_{\omega_2}\Phi_\omega^* \rangle\,, \tag{2}$$

$$\langle \Phi_\omega(\tau)\Phi_{\omega_1'}^*\Phi_{\omega_2'}^* \rangle = L_\omega \langle \Phi_\omega \Phi_{\omega_1'}^*\Phi_{\omega_2'}^* \rangle + \frac{1}{2}\sum Q_{\omega_1,\omega_2}\langle \Phi_{\omega_1}\Phi_{\omega_2}\Phi_{\omega_1'}^*\Phi_{\omega_2'}^* \rangle\,, \tag{3}$$

where τ is a time delay and $L_\omega = (\gamma_k + i\omega_k)\tau + 1 - i[\theta(k,t+\tau) - \theta(k,t)]/e^{-i[\theta(k,t+\tau)-\theta(k,t)]}$ and $Q_{\omega_1,\omega_2} = [\Lambda_k^Q(k_1,k_2)\tau]/(e^{-i[\theta(k,t+\tau)-\theta(k,t)]})$ can be solved from experimental data. One prerequisite of this approach is obtaining the fourth order moment $\langle \Phi_{\omega_1}\Phi_{\omega_2}\Phi_{\omega_1'}^*\Phi_{\omega_2'}^* \rangle$, which is computationally intensive. This is solved by Ritz *et al.* by approximating the fourth order moment with second order moments $\langle |\Phi_{\omega_1}\Phi_{\omega_2}|^2 \rangle$ by neglecting terms $(\omega_1', \omega_1') \neq (\omega_1, \omega_2)$. On the Columbia Linear Machine (CLM) [2], which produces a collisionless hydrogen plasma with a density N of 5×10^8 cm^{-3}, electron temperature T_e of 5 eV, and ion temperature T_i of 3 eV, we are in an excellent position to perform this kind of experiments. One major advantage is due to the steady state plasma that our device produces, which allows us to obtain good ensemble averaging. Also, since the fluctuations in CLM are mostly narrow band with distinct frequency peaks, one can easily identify discrete three wave coupling triplets with distinct k numbers, as opposed to the broad band turbulence present in many other devices. Furthermore, in the past we have performed several experiments using feedback from an ion/electron beam source [3,4], or even a Langmuir probe, which enabled us to suppress/enhance the underlying linear drive of the instability [5]. In fact, in this experiment we make novel exploitation of this feedback system to avoid calculating the fourth order moments, as well as generating richer data sets. We include linear feedback in Eq. (1) by adding $G\phi(k,t)$ on the right-hand side, where G is the complex (constant) gain of the feedback loop. It can be seen that G modifies only the linear operator of Eq. (1), such that Eq. (2) can be rewritten as

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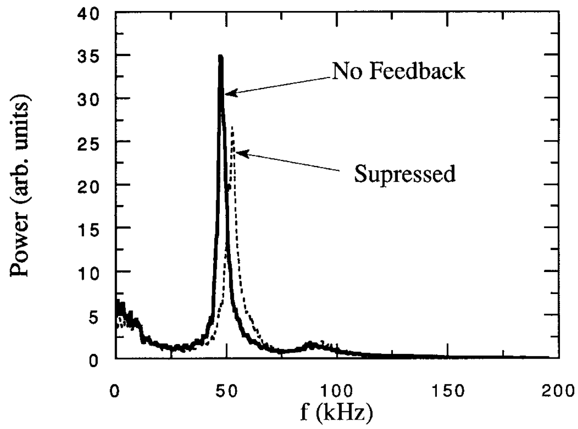

FIG. 1. Power spectrum of plasma instabilities present in CLM. Two cases are provided, one with moderate feedback suppression and one with feedback turned off.

$$\langle \Phi_\omega(\tau)\Phi_\omega^* \rangle' = \left(L_\omega + \frac{G}{e^{-i[\theta(k,t+\tau)-\theta(k,t)]}} \right) \langle \Phi_\omega \Phi_\omega^* \rangle' + \frac{1}{2} \sum Q_{\omega_1,\omega_2} \langle \Phi_{\omega_1} \Phi_{\omega_2} \Phi_\omega^* \rangle'. \quad (4)$$

By experimentally adjusting for different feedback gain G, and measuring the subsequent fluctuation signal, we can generate an arbitrary number of equations with a fixed number of unknowns. This, in principle, allows us to solve for $\gamma_k + i\omega_k$ and the coupling coefficients $\Lambda_k^Q(k_1, k_2)$ for each particular mode k.

The mode used for this experiment is a centrifugal flute mode driven by the $\mathbf{E} \times \mathbf{B}$ rotation of the plasma [6], whose power spectrum is shown in Fig. 1. The radial electric field necessary for the rotation is naturally born in most plasmas and can be enhanced in our machine via end plate bias. For sufficient levels of equilibrium rotation, a dominant mode is driven along with two harmonics. Also shown in the figure is the mode behavior after moderate suppression using feedback. Under most conditions, the dominant mode has azimuthal mode number $m = 1$, with typical frequency $f = 50$ kHz and a broad radial extent equivalent to an $n = 0$ radial harmonic. Also present, but usually at a few kHz higher than the dominant mode, is a mode which shows a radial structure consistent with an $n = 1$ harmonic, as previously discovered and discussed in [7]. Even though the $n = 1$ radial harmonic is not directly evident from the frequency spectrum, its existence becomes apparent when the three wave coupling triplets are being identified. This can be seen from the bicoherence contour plot shown in Fig. 2, which measures the second order correlation of three waves with frequency f_1, f_2, and $f_1 + f_2$, where f_1 and f_2 are plotted on the x and the y axis, respectively. Based on the number of ensemble averages taken and noise present in the fluctuation, correlation higher than 0.1 can be taken to be of significance for this experiment. For reason of clarity of display, the bicoherence in Fig. 2 represents the case for strongest coupling, i.e., maximum enhancement during feedback. For all other cases, the bicoherence is reduced accordingly, sometimes to the point where the bicoherence falls below 0.1. However, even in these cases it is assumed that the coupling is still present, albeit weaker. Several sets of coupling triplets can be identified from the plot. One set indicates harmonic generation of azimuthal modes, i.e., the self-coupling of the mode at frequency f with itself and $2f$, indicated by the strong peak at the diagonal of the bicoherence plot. Also, one can see the dominant mode coupling with other modes at around 80–100 kHz, indicated by a series of weaker peaks along $x = 80$–100 kHz, $y = 48$–52 kHz. More interestingly is the coupling of the dominant mode with the higher radial harmonic. This can

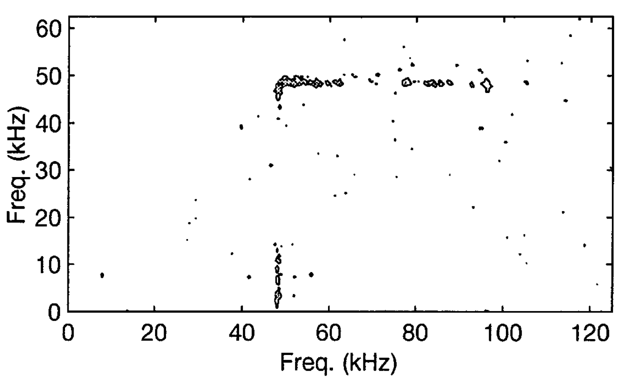

FIG. 2. Bicoherence contour plot of the plasma instability. Each contour step corresponds to bicoherence of 0.06. Because of the symmetry properties, only the lower diagonal half is displayed. Peaks on the diagonal represent self-coupling to its harmonic, whereas peaks offset from the diagonal represent coupling of two separate modes.

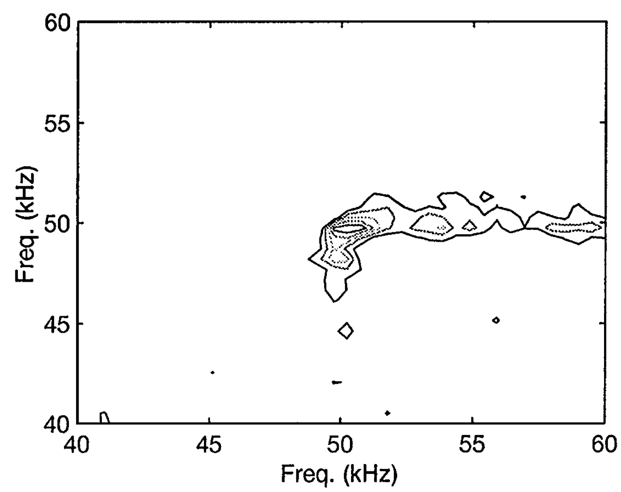

FIG. 3. Close-up of Fig. 2. The contour extending off the diagonal represents coupling of the dominant mode with a "radial" harmonic close by in frequency. Each contour line represents an increase of value by 0.06.

be seen from the close-up view of the previous contour plot, shown in Fig. 3. Here, the peak on the diagonal, which represents the self-coupling, is being extended along a fixed frequency, which indicates a coupling triplet of modes at f, $(f + \Delta f)$, and $(2f + \Delta f)$. Further evidence is present by observing the coupling of these two modes into a difference mode, i.e., a coupling triplet at Δf, f, and $(f + \Delta f)$, shown as peaks in Fig. 2 at $x = 48-52$ kHz and $y = 3-8$ kHz.

First, we estimate the real frequencies ω_k from bicoherence and power spectrum. Then, to determine the growth rate and coupling coefficients, only the real part of Eq. (4) is used, since it is numerically more stable to do so. Equation (4) is further modified by letting $e^{i[\theta(k,t+\tau)-\theta(k,t)]} \simeq \langle \Phi_\omega(\tau)\Phi_\omega^*\rangle / |\langle \Phi_\omega(\tau)\Phi_\omega^*\rangle| \simeq e^{i\omega\tau}$:

$$|\langle \Phi_\omega(\tau)\Phi_\omega^*\rangle'| = [(\gamma_k - |G|\cos\theta)\tau + 1\langle \Phi_\omega\Phi_\omega^*\rangle'] + \mathrm{Re}\left(\frac{1}{2}\sum \Lambda_k^Q(k_1,k_2)\tau\langle \Phi_{\omega_1}\Phi_{\omega_2}\Phi_\omega^*\rangle'\right). \quad (5)$$

All the above quantities related to the fluctuation signal Φ_ω are obtained by measuring the density fluctuation using a Langmuir probe biased to collect ion saturation current, and then performing the appropriate mathematical functions once the signal was digitized to 12 bits accuracy. Since all the modes have some spectral width, the power of the modes is integrated over the appropriate frequency band, after they have been ensemble averaged over 200 samples. The feedback setup is almost identical as described in [5], except that the actuator/suppressor consists of a separate Langmuir probe for convenience, even though our ion/electron beam provides similar results. Even though the suppressor probe is localized, it works via excitation of the eigenmodes of the device which is a global flute mode for a certain set of plasma parameters. The physics is similar to the excitation of vibrational eigenmodes of a violin string by striking it with a delta function force. Direct experimental determination of the gain and phase of the feedback loop must be done under closed loop conditions to which these pertain. Hence, for the purpose of this experiment, the magnitude of the gain is set to a fixed level, such that it still provides good suppression/enhancement. Then, to determine this fixed gain, the phase of the closed loop is varied using a time delay circuit. As this "phase shifter" is varied through 180°, not only can one observe the suppression/enhancement of the mode, but one can also observe the mode shifting in frequency, evident in Fig. 1. According to our model, at the point when the gain is entirely imaginary, its contribution should be only in shifting the real frequency without affecting the instability's growth rate. At this phase shift setting, the magnitude of the gain then directly corresponds to the mode shift in frequency, which can be directly measured from the experiment. This phase shift setting then also serves as a calibration point for the phase shifter ($\theta = 90°$). After this calibration, the magnitude of the gain is kept constant, and the phase is adjusted to obtain the different sets of equations. In order to avoid solving for excessively insignificant coupling coefficients, some triplets are truncated or omitted. This includes all triplets where two out of the three modes exhibit low power, such as the coupling of the dominant mode with the mode at 80–100 kHz to a third mode at above 120 kHz. This is possible because for the actual calculation the bispectrum is needed instead of the normalized bicoherence, such that low mode amplitude results in low bispectrum. Also, in theory the self-coupling of the mode at f and the coupling with the radial harmonic at $(f + \Delta f)$ couples to two independent modes at $2f$ and $(2f + \Delta f)$, respectively. In reality, it is impossible to accurately distinguish these two modes, as the mode power at these frequencies again is quite small. Therefore, these two modes at $2f$ and $(2f + \Delta f)$ are lumped together and considered as a "single" mode. Finally, one should note that since the bispectrum is needed instead of the bicoherence, in most cases high bispectrum values are obtained despite low bicoherence results. Hence, as long as the coupling triplets can be correctly identified from one of the cases, low bicoherence in other cases do not necessarily indicate poor bispectrum results.

It is found that the bicoherence readings are far better during feedback enhancement than during suppression, mostly due to the increased signal to noise ratio from enhancement. Therefore, data are taken at seven different phase settings, all of which provide enhancement of the mode. These, along with the case of not applying any feedback, provide eight independent equations. They are then used to solve for all the unknowns. Since even the dominant mode has the highest number of only seven unknowns, singular value decomposition is used to solve for the unknowns. The result is summarized in Table I and discussed below. First, it is noted that the dominant mode k has a positive growth rate, while the two other possible members of a triad have negative growth (damping) rates. This is certainly consistent with the physics requirement that for the nonlinear saturation of a growing mode one needs coupling to damped modes which provide the energy sinks for the establishment of a steady state. In the above scenario, we single out a triplet (k, k_1, k_3) as the dominant three wave coupling mechanism in our experiment, based on the fact that the amplitude of the k_2 mode is the weakest. Furthermore, it is involved jointly in a three wave process, as well as harmonic generation, making it very difficult to discern and isolate the two experimentally. Approximate estimates of the real frequencies and growth rates have been theoretically obtained as follows [8]: $f \sim 60$, $\gamma_k \sim 1$ kHz; $f_{k_1} \sim 72$, $\gamma_{k_1} \sim -2.3$ kHz; $f_{k_2} \sim 142$,

TABLE I. Table showing growth rate γ (Hz) and coupling coefficient Λ (Vs^{-1}) related to each mode.

For mode k: $f = 50$ kHz, $m = 1$, $n = 0$ flute mode:				
$\gamma_k = 2100$,	$\mathrm{Re}\{\Lambda_k^Q(k,k_2)\} = 560$,	$\mathrm{Im}\{\Lambda_k^Q(k,k_2)\} = 160$,	$\mathrm{Re}\{\Lambda_k^Q(k_1,k_2)\} = -3700$,	$\mathrm{Im}\{\Lambda_k^Q(k_1,k_2)\} = 6700$,
	$\mathrm{Re}\{\Lambda_k^Q(k_1,k_3)\} = -120$,	$\mathrm{Im}\{\Lambda_k^Q(k_1,k_3)\} = 940$		
For mode k_1: $f = 55$ kHz, $m = 1$, $n = 1$ flute mode:				
$\gamma_{k_1} = -2500$,	$\mathrm{Re}\{\Lambda_{k_1}^Q(k,k_2)\} = -2800$,	$\mathrm{Im}\{\Lambda_{k_1}^Q(k,k_2)\} = 940$,	$\mathrm{Re}\{\Lambda_{k_1}^Q(k,k_3)\} = -620$,	$\mathrm{Im}\{\Lambda_{k_1}^Q(k,k_3)\} = 630$
For mode k_2: $f \simeq 102$ kHz, $m = 2$, $n = 0$ flute mode:				
$\gamma_{k_2} = -17\,000$,	$\mathrm{Re}\{\Lambda_{k_2}^Q(k,k)\} = 150$,	$\mathrm{Im}\{\Lambda_{k_2}^Q(k,k)\} = 270$,	$\mathrm{Re}\{\Lambda_{k_2}^Q(k,k_1)\} = -630$,	$\mathrm{Im}\{\Lambda_{k_2}^Q(k,k_1)\} = -3500$
For mode k_3: $f \simeq 5$ kHz, drift wave:				
$\gamma_{k_3} = -910$,	$\mathrm{Re}\{\Lambda_{k_3}^Q(k,k_1)\} = 400$,	$\mathrm{Im}\{\Lambda_{k_3}^Q(k,k_1)\} = -80$		

$\gamma_{k_2} \sim -6$ kHz. With the exception of γ_{k_2}, these indicate rough agreement with experimental results presented in Table I. The reason for the large discrepancy in γ_{k_2} is clearly attributable to the small signal to noise ratio due to the small amplitude of this mode. It is also noted that there is about a 7% frequency mismatch in the theoretical resonance condition which is quite plausible.

Even with a steady state plasma and good ensemble averaging, the consistency of the bispectrum (and bicoherence) results are still less than desired. One possible explanation is that there are long term drifts in our plasma device, which become apparent when many averages need to be taken over a long period of time. This affects mostly the accuracy of the coupling coefficients, and not the growth rate. The growth rate, however, is more dependent on the estimation of the magnitude of the gain. If the gain magnitude is overestimated, then the growth rates will also be systematically higher. The assumption that feedback modifies only the linear operator is reasonable, but not provable in general. In this case it is confirmed by monitoring the mapping between the feedback phase shifts and the frequency shifts of the mode, which follows the model closely.

In general, linear feedback does not change the equilibrium parameters of a system and is therefore noninvasive. In our experiment we have clear evidence that feedback does not alter the plasma parameters like density, electron and ion temperatures, and electric potentials and their gradients, which determine the frequencies, growth (damping) rates, wave numbers, and, consequently, nonlinear coupling coefficients. A model of three wave nonlinear coupling can be supported on the basis that other possible nonlinear mechanisms of mode saturation in our experiment are not viable. These include quasilinear flattening of profiles which is not observed experimentally or nonlinear Landau damping which is not possible for flute modes discussed here. The plasma modes discussed here are variations of drift modes which have been observed in tokamaks, stellarators, and other plasma devices in various guises and are strongly believed to be responsible for the largely intractable question of anomalous transport. In summary, the method described here is a novel approach of significant promise for the determination of plasma turbulence models. Compared to the other theoretical methods, it makes direct use of experimental data as much as possible and avoids calculating higher order moments, unlike a recent theoretical/experimental method using a modified and improved Ritz method [9]. Furthermore, because of the use of continuously variable feedback, reliable results from rich data sets are possible.

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