Oscillatory Null Singularity inside Realistic Spinning Black Holes

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We calculate the asymptotic behavior of the curvature scalar (Riemann)^2 near the null weak singularity at the inner horizon of a generic spinning black hole, and show that this scalar oscillates an infinite number of times while diverging. The dominant parallel-propagated Riemann components oscillate in a similar manner. This oscillatory behavior, which is a remarkable contrast to the monotonic mass-inflation singularity in spherical charged black holes, is caused by the dragging of inertial frames due to the black hole's spin.

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One of the major challenges in classical general relativity during the past decades has been to explore the nature of the spacetime singularities which form in gravitational collapse. The existence of singularities inside black holes has been verified by several mathematical theorems [1]. However, the singularity theorems do not tell us much about the location and features of these singularities.

It is widely anticipated that the realistic astrophysical black holes are rapidly spinning [2,3]. The simplest type of a spinning black hole (BH) is given by the Kerr solution [4], describing a stationary, axially symmetric, spinning vacuum BH. The Penrose diagram of the Kerr geometry is displayed in Fig. 1. The inner horizon (IH) is a null hypersurface located inside the BH. This hypersurface, also known as the Cauchy horizon [1], marks the boundary of predictability for physical fields whose initial data are specified outside the BH. In the pure Kerr geometry, the IH is a perfectly smooth surface. Penrose [5] pointed out, however, that ingoing electromagnetic or gravitational perturbations are infinitely blueshifted at the IH. He therefore suggested that in a more realistic BH, which is not strictly stationary, the infinitely blueshifted perturbations will lead to the formation of a curvature singularity instead of a regular IH (we shall refer to this singularity as the *IH singularity*). The instability of the inner horizon was later investigated by several authors, who used a spherical charged BH as a toy model [6] (this is a useful toy model, because a spherical charged BH also admits an inner horizon with infinite blueshift). A few analyses of linear fields inside a Kerr BH were also carried out at the end of the 1970s [7,8]. (For recent analyses of the late-time behavior of gravitational perturbations *outside* a Kerr BH, see Ref. [9].)

About ten years ago, in an effort to explore the nonlinear aspects of the IH singularity, Poisson and Israel [6] introduced the mass-inflation model—a spherically symmetric model made of a charged BH with two radial null fluids (ingoing and outgoing). In this model they obtained a null curvature singularity at the IH, known as the *mass-inflation singularity*. This singularity is marked by an exponential growth of curvature. A more detailed study [10] later revealed that the mass-inflation singularity is weak in Tipler's [11] terminology. Namely, physical objects only experience finite tidal distortion when they approach the singularity.

Later, Ori [12] investigated the geometry inside a realistic spinning BH using a perturbative approach (see also [13]). This analysis revealed that in the spinning case, too, there is a null, weak, curvature singularity at the IH. The main results of the perturbative analysis [12] were later confirmed by several nonperturbative local analyses [14–17].

In general, the features of the IH singularity of spinning BHs are found to be very similar to that of spherical charged BHs: In both cases, the singularity is null, weak, and blueshift dominated. There is one important difference, however: The mass-inflation singularity is characterized by *a monotonic* growth of the mass function (and curvature) [6,10]. However, the IH singularity of a spinning BH is *oscillatory*, as we shall show in this

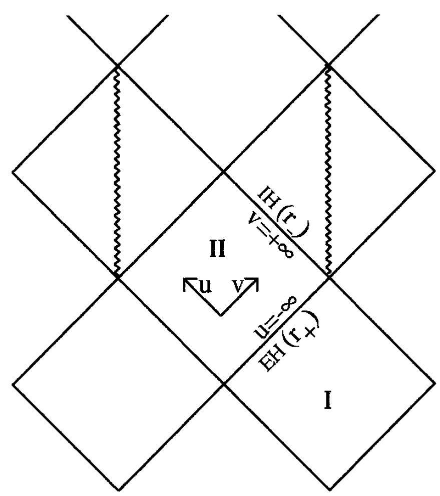

FIG. 1. Penrose diagram of the extended Kerr geometry. Region I is the external universe. Region II is located inside the BH, between the EH and the IH. This figure also displays the range of the coordinates u, v in region II.

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paper. This oscillatory behavior is related to the dragging of inertial frames due to the BHs spin.

One of the rather surprising findings of the perturbation analysis [12] is that the IH singularity is essentially linear. Namely, at the early portion of the IH, the structure of the singularity may adequately be described (at the leading order) by the linear gravitational perturbation over the Kerr background, because the effect of higher-order nonlinear perturbation terms is negligible. Motivated by this observation, we have recently carried out a detailed analysis [18] of linear gravitational perturbations over the Kerr background, using the Newman-Penrose (NP) formalism. Based on the results of this analysis (along with that of Ref. [12]), we shall now calculate the asymptotic behavior of the curvature at the IH singularity and reveal its oscillatory character. For concreteness, we shall focus on the quadratic curvature scalar $K \equiv R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$. We shall consider a nonextreme, pure vacuum BH, and will restrict attention to the early portion of the IH singularity (where the perturbation analysis [12] is effective).

The event horizon (EH) and the IH of the background Kerr geometry are located at the hypersurfaces $r = r_+$ and $r = r_-$, respectively, where $r_\pm \equiv M \pm (M^2 - a^2)^{1/2}$. Here M and a denote the BHs mass and specific angular momentum, respectively. We use here the Boyer-Lindquist [19] coordinates (t, r, θ, φ). The Eddington-like coordinates u, v are given by $v = r^* + t$ and $u = r^* - t$, where $r^*(r)$ is defined by $dr/dr^* = \Delta/(r^2 + a^2)$ and $\Delta \equiv (r - r_+)(r - r_-)$. The event horizon and the IH correspond to $u = -\infty$ and $v = \infty$, respectively (see Fig. 1).

Following Ref. [12], we express the metric $g_{\alpha\beta}$ of the perturbed spinning BH as the sum of the unperturbed Kerr metric and the metric perturbation $h_{\alpha\beta}$. The latter is then expanded in the form

$$h_{\alpha\beta} = h^{(1)}_{\alpha\beta} + h^{(2)}_{\alpha\beta} + h^{(3)}_{\alpha\beta} + \dots, \qquad (1)$$

where $h^{(1)}_{\alpha\beta}$ is the linear metric perturbation, $h^{(2)}_{\alpha\beta}$ is the second-order perturbation, etc. Here we adopt the gauge used in Ref. [12], in which all terms $h^{(J)}_{\alpha\beta}$ are finite at the IH (and are arbitrarily small at its early portion), and the null curvature singularity is located strictly at $r = r_-$ (i.e., at the IH) of the Kerr background. This singularity is marked by the divergence of the curvature scalar K. Note that K (like the Riemann tensor itself) is perfectly regular at the IH of the unperturbed Kerr background, and its divergence in a realistic spinning BH is caused by the gravitational perturbation, which is infinitely blueshifted at the IH.

Comparing the asymptotic forms of the various terms $h^{(J)}_{\alpha\beta}$, one finds that $h_{\alpha\beta}$ is dominated by the linear perturbation $h^{(1)}_{\alpha\beta}$ [12]; The higher-order terms are smaller by certain powers of $1/v$ and/or $1/u$ (which is arbitrarily small in the early portion of the IH). As a consequence, it is not difficult to show that K is dominated by $\hat{K} \equiv \hat{R}_{\alpha\beta\gamma\delta}\hat{R}^{\alpha\beta\gamma\delta}$, where $\hat{R}_{\alpha\beta\gamma\delta}$ denotes the linear perturbation in the Riemann tensor. In what follows we shall use the NP formalism to calculate $\hat{K}$.

In a vacuum spacetime, the Riemann tensor may be expressed as a linear combination of the five NP Weyl scalars Ψ_i $(i = 0, \dots, 4)$ and their complex conjugate (see, e.g., Eq. (1.298) in [20]). We schematically write this linear combination as

$$R_{\alpha\beta\gamma\delta} = Q^i_{\alpha\beta\gamma\delta}\Psi_i + \text{c.c.}, \qquad (2)$$

where $Q^i_{\alpha\beta\gamma\delta}$ are constants, and c.c. denotes the complex conjugate. Explicitly calculating these constants according to the method explained in Ref. [20], and then squaring the last equation, one finds

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = 8(\Psi_0\Psi_4 + 3\Psi_2^2 - 4\Psi_1\Psi_3) + \text{c.c.} \qquad (3)$$

In a similar manner, by picking the linear perturbations of the quantities in both sides of Eq. (2) and squaring them, one obtains an analogous expression for $\hat{R}_{\alpha\beta\gamma\delta}$:

$$\hat{K} \equiv \hat{R}_{\alpha\beta\gamma\delta}\hat{R}^{\alpha\beta\gamma\delta} \cong 8(\psi_0\psi_4 + 3\psi_2^2 - 4\psi_1\psi_3) + \text{c.c.}, \qquad (4)$$

where ψ_i denotes the linear perturbation in Ψ_i. [We have ignored here all contributions proportional to the (undifferentiated) metric perturbation $h_{\alpha\beta}$, e.g., those obtained when indices are raised or lowered. These turn out to be negligibly small, like $h_{\alpha\beta}$ itself.]

From the asymptotic expressions for the linear metric perturbations [12], one can evaluate the maximal possible divergence rates of the various linear NP fields at the IH. One can show that the maximal inverse powers of $r - r_-$ involved in this divergence are [21]

$$\psi_0 \propto (r - r_-)^{-2}, \qquad \psi_1 \propto (r - r_-)^{-1}, \qquad \psi_{2,3,4} \propto (r - r_-)^0. \qquad (5)$$

Therefore,

$$K \cong \hat{K} \cong 8\psi_0\psi_4 + \text{c.c.} \qquad (6)$$

This result is remarkable for two reasons. First, ψ_0 and ψ_4 are gauge invariant [20] [whereas $\psi_{1,2,3}$ are not; the expressions for $\psi_{1,2,3}$ in Eq. (5) are obtained in the gauge used in Ref. [12] and here]. Second, both ψ_0 and ψ_4 satisfy a simple master equation [22]. [Note: In the gauge we use, Ψ_2 is dominated by its second-order term $\Psi_2^{(2)}$, which diverges like $(r - r_-)^{-1}$ [whereas $\Psi_2^{(1)} \equiv \psi_2 \propto (r - r_-)^0$]. Nevertheless, the contribution of $\Psi_2^{(2)}$ (squared) to K is smaller than $\hat{K}$ by certain powers of $1/u$ or $1/v$, as was mentioned above (with regards to the contribution of nonlinear perturbations to K). All other NP fields Ψ_i $(i \neq 2)$ are dominated by their linear counterparts ψ_i.]

The evolution of ψ_0 and ψ_4 inside a Kerr BH was analyzed in Ref. [18]. [Note that $\Psi^{s=2}$ and $\Psi^{s=-2}$ therein correspond, respectively, to ψ_0 and $(r - ia\cos\theta)^4\psi_4$ in the notation of the present paper.] For generic initial data, one finds that both ψ_0 and ψ_4 are dominated by the modes with $l = 2$ (which have the slowest decay rate, t^{-7}, outside the BH). The asymptotic behavior of ψ_4 at the early portion of the IH is found to be (see section IX in Ref. [18])

$$\psi_4 \cong u^{-8}(r_- - ia\cos\theta)^{-4} \times \sum_{m=-2}^{2} A_m \, {}_{-2}Y_2^m(\theta,\phi) e^{-im\Omega_- u} + O(u^{-9}), \quad (7)$$

where $\Omega_- \equiv a/(2Mr_-)$, $\phi \equiv \varphi - \Omega_- t$ is an azimuthal coordinate regular at the IH [20], and ${}_sY_l^m$ denotes the spin-weighted spherical harmonics. The asymptotic behavior of ψ_0 is

$$\psi_0 \cong (r - r_-)^{-2} v^{-7} \sum_{m=-2}^{2} B_m \, {}_2Y_2^m(\theta,\phi) e^{im\Omega_- v} + O(v^{-8}). \quad (8)$$

A_m and B_m are coefficients which are proportional to the initial amplitudes of the modes ($l = 2$, $|m| \le 2$) of ψ_4 and ψ_0, respectively. These coefficients are generically nonvanishing. The only exception is the coefficient B_0, which vanishes identically (that is, the mode $l = 2$, $m = 0$ of ψ_0 is $\propto (r - r_-)^{-2} v^{-8}$ at the IH; see Ref. [18]). Substituting Eqs. (7) and (8) into Eq. (6), we obtain

$$K \cong (r - r_-)^{-2} v^{-7} \sum_{m=1,2} C_m(u,\theta,\phi) e^{im\Omega_- v} + \text{c.c.}, \quad (9)$$

with (generically) nonvanishing coefficients C_m. Note that no $m = 0$ term is present at the leading order, due to the vanishing of B_0.

Consider now a freely falling observer which hits the IH singularity at a point (u_0, θ_0, ϕ_0). For this observer, $r - r_-$ and v are proportional to τ and $\ln(-\tau/M)$, respectively, where τ denotes the proper time, and we have set $\tau = 0$ at the intersection with the IH singularity. One obtains

$$K \cong c\tau^{-2}[\ln(-\tau/M)]^{-7} \times \sum_{m=1,2} C_m(u_0,\theta_0,\phi_0) e^{-imp\ln(-\tau/M)} + \text{c.c.}, \quad (10)$$

where c is a nonvanishing constant that depends on the geodesic's constants of motion, and $p \equiv a(M^2 - a^2)^{-1/2}$. In a similar manner, one finds that the most divergent components of the Riemann tensor (as measured by a parallel-propagated tetrad) are $\propto \psi_0$, and are hence proportional to

$$\tau^{-2}[\ln(-\tau/M)]^{-7} \sum_{m=1,2} c_m \, {}_2Y_2^m(\theta_0,\phi_0) e^{-imp\ln(-\tau/M)} + \text{c.c.} \quad (11)$$

with nonvanishing constants c_m.

From Eq. (10) it is obvious that, while diverging like τ^{-2} (softened by an inverse-power logarithmic factor), the curvature scalar K undergoes an infinite number of oscillations. In particular, K vanishes and changes sign infinitely many times on the approach to the IH singularity. The dominant parallel-propagated Riemann components, given in Eq. (11), behave in a similar manner. Thus, the IH singularity of a generic spinning BH is oscillatory.

This oscillatory behavior is in remarkable contrast to the monotonic increase of the mass function (and curvature) in the mass-inflation singularity of spherical charged BHs. The oscillations are caused by the dragging of inertial frames, due to the BHs angular momentum. More specifically, the dragging of the nonaxially symmetric modes (which dominate ψ_0) leads to oscillations in v.

It has been argued by Belinsky, Khalatnikov, and Lifshitz (BKL) [23] that a generic singularity (the *BKL singularity*) exists in the solutions of the vacuum Einstein equations which is spacelike and oscillatory. Recent numerical and analytical investigations provide further evidence for the existence of such singular vacuum solutions [24]. It is remarkable that both known generic singularities—the BKL singularity and the spinning inner-horizon singularity—are oscillatory. Note, however, that apart from this common nonmonotonic character, these two singularities are very different from each other: The BKL singularity is spacelike, strong, and extremely complicated (perhaps even chaotic), whereas the inner-horizon singularity is null, weak, and of a rather simple asymptotic form.

There is also an important difference in the status of these two singularities in connection with their actual occurrence in realistic gravitational collapse (or, at least, in connection with our present knowledge about their actual occurrence). The actual formation of the null weak inner-horizon singularity in a generic gravitational collapse has been verified in an explicit manner by the perturbative analyses [12,13,18]. (The local consistency and genericity of this singularity have been verified also by several nonperturbative local analyses [14–17].) On the other hand, the analyses of the BKL singularity indicated the local consistency of this singularity, and probably also its inevitable occurrence in certain cosmological models, but so far not in asymptotically flat situations. There certainly exist generic asymptotically flat initial-data sets which do not develop a BKL singularity (e.g., any set with a sufficiently weak initial field, such that no black hole forms). One may attempt to conjecture that generically any asymptotically flat initial-data set which develops a black hole will also develop a BKL singularity inside it, but we are not aware of any compelling evidence for such a conjecture (recall also that the predictive power of the singularity theorems is exhausted by the null inner-horizon singularity, which definitely forms in a generic collapse). In fact, for reasons which are beyond the scope of this paper, I believe that the above conjecture is incorrect (but a weaker version, which puts restrictions on the spatial topology, may be attempted).

Recent numerical studies of spherical charged BHs perturbed by a self-gravitating scalar field indicate that a generic spacelike singularity forms when the area of the mass-inflation singularity shrinks to zero [25]. It is unclear, however, whether an analogous spacelike singularity will form in realistic spinning black holes, which are nonspherical, and which have no scalar field. Note that the above scalar-field spacelike singularity is monotonic [26]. This type of generic spacelike singularity probably has no counterpart in vacuum spacetimes: Both the original work by BKL [23] and the recent analyses by Berger and collaborators [24] suggest that there exists no generic, monotonic, spacelike, vacuum singularity. Therefore, although there is a likelihood that a BKL-like spacelike singularity will form inside realistic spinning black holes, this is still far from obvious, and further research is required in order to clarify this issue.

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