Hydrodynamics of the Kuramoto-Sivashinsky Equation in Two Dimensions

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The large scale properties of spatiotemporal chaos in the 2D Kuramoto-Sivashinsky equation are studied using an explicit coarse-graining scheme. A set of intermediate equations are obtained which describe interactions between the small scale structures and the hydrodynamic degrees of freedom. Possible forms of the effective large scale hydrodynamics are constructed and examined. Although a number of different universality classes are allowed by symmetry, numerical results support the simplest scenario, that being the Kardar-Parisi-Zhang universality class.

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A major goal in the study of spatiotemporal chaos (STC) [1] is to obtain quantitative connections between the chaotic dynamics of a system at small scales and the apparent stochastic behavior at large scales. The Kuramoto-Sivashinsky (KS) equation [2]

$$\partial_t h = -\nabla^2 h - \nabla^4 h + (\nabla h)^2 \tag{1}$$

has been used as a paradigm in efforts to elucidate the micro-macro connections [3,4].

The qualitative behavior of the KS equation is quite simple. Cellular structures are generated at scales of the order $\ell_0 = 2\sqrt{2}\,\pi$ due to the linear instability. These cells then interact chaotically with each other via the nonlinear spatial coupling to form the STC steady state at scales much larger than ℓ_0. The characterization of the STC state has been studied extensively in one spatial dimension [3–6]. It was conjectured by Yakhot [3], based partially on symmetry grounds, that the large scale behavior of the one-dimensional KS (1D-KS) equation is equivalent to that of the 1D noisy Burgers equation, also known as the Kardar-Parisi-Zhang (KPZ) equation [7]. This conjecture has since been validated by detailed numerical studies [4,5]. More recently, an explicit coarse-graining procedure was used by Chow and Hwa [6] to derive a set of coupled effective equations describing the interaction between the chaotic cellular dynamics and the long wavelength fluctuations of the h field. From this description, the large scale (KPZ-like) behavior of the 1D-KS system can be predicted quantitatively from the knowledge of various response functions at the "mesoscopic scale" of several ℓ_0's.

The behavior of the 2D-KS equation is not as well understood. The simplest scenario is the generalization of Yakhot's conjecture to 2D, with the large scale behavior described by the 2D-KPZ equation,

$$\partial_t h = \nu\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(\mathbf{r},t), \tag{2}$$

where $\nu > 0$ can be interpreted as a stabilizing "surface tension" for the height profile h, and η as a *stochastic* noise with $\langle\eta(\mathbf{r},t)\eta(\mathbf{r}',t')\rangle = 2D\,\delta^2(\mathbf{r}-\mathbf{r}')\delta(t-t')$. For $\nu > 0$, the asymptotic scaling properties of Eq. (2) are described by "strong-coupling" behavior with algebraic (rather than logarithmic) scaling in the roughness of h and superdiffusive dynamics [8]. The length scale at which the asymptotic regime is reached is given by $\ell_\times \sim e^{8\pi/g}$, where $g \equiv \lambda^2 D/\nu^3$. At scales below $\ell_\times$, the effect of the nonlinear term in (2) can be accounted for adequately via perturbation theory. The system behaves in this "weak-coupling" regime as a linear stochastic diffusion equation with additive logarithmic corrections [9].

Previous studies of the 2D-KS equation [10–12] found behavior consistent with linear diffusion with logarithmic corrections but had different interpretations. Jayaprakash *et al.* [12] performed a numerical analysis akin to Zaleski's on the 1D-KS [4], and concluded that their results were consistent with the weak-coupling regime of the 2D-KPZ equation, with (in principle) a crossover to strong coupling beyond a length of $\ell_\times \approx 10^{26}\ell_0$, for $g = 0.4$. Procaccia *et al.* [10,11] used a comparative Dyson-Wyld diagrammatic analysis of the two equations to argue that 2D-KS and 2D-KPZ *cannot* belong to the same universality class [13]. They maintained instead that the asymptotic behavior of the 2D-KS equation is described by a "nonlocal" solution, consisting of diffusion with *multiplicative* logarithmic corrections [14]. We feel that the ensuing debate [15] failed to rule out either interpretation.

It is very difficult to distinguish between the above two scenarios numerically, as one must resolve different forms of logarithmic corrections to the (already logarithmic) correlation function of the linear diffusion equation. Theoretically, there is no *a priori* reason why simple symmetry considerations such as Yakhot's should be valid in two and higher dimensions. Unlike in 1D where there are only scalar density fluctuations, the 2D case is complicated because three or more large-k modes can couple and contribute to low-k fluctuations. Such nonlocal interactions in k may not be adequately accounted for in the type of analysis performed in Refs. [4,12], which numerically impose KPZ dynamics and then test for self-consistency.

In this paper, we perform a systematic symmetry analysis, taking into account the possibility of large-k

coupling. Specifically, we extend the coarse-graining procedure of Ref. [6] to two dimensions to derive a set of coupled equations describing the local arrangement of cells, and study their effect on the macroscopic dynamics of the h field. The resulting behavior depends crucially on the small scale arrangement of the cells. In the simplest case, the strong-coupling 2D-KPZ behavior is recovered. Nevertheless, more complicated behaviors are allowed if the microscopic cellular arrangement exhibits *spontaneous rotational symmetry breaking*. A number of possible scenarios are listed for this case. To determine which of the allowed scenarios is selected by the 2D-KS equation, we performed numerical measurements of the cellular dynamics at the mesoscopic scale of 4 to 16 ℓ_0's. Our results disfavor the occurrence of the more exotic scenarios, leaving the strong-coupling 2D-KPZ behavior as the most likely possibility.

As in 1D, we coarse grain over a region of size $L \times L$, where L is several times the typical cellular size ℓ_0. $h(\mathbf{r}, t)$ is separated into fast cellular modes $h_>$ and slow long wavelength modes $h_<$. Inserting $h(\mathbf{r}, t) = h_<(\mathbf{r}, t) + h_>(\mathbf{r}, t)$ into Eq. (1), we obtain the following equations for the fast and slow modes:

$$\partial_t h_> = -\nabla^2 h_> - \nabla^4 h_> + (\nabla h_>)^2_> + 2(\nabla h_> \cdot \nabla h_<), \tag{3}$$

$$\partial_t h_< = -\nabla^2 h_< + (\nabla h_<)^2 + w(\mathbf{r}, t) + O(\nabla^4 h_<). \tag{4}$$

where $w(\mathbf{r}, t) \equiv (\nabla h_>)^2_<$ is the only contribution of the fast modes on the dynamics of $h_<$. It can be interpreted as the "drift rate" of $h_<$ over a regime of $L \times L$ centered at $\mathbf{r}$. To specify the dynamics of $h_<$, it is necessary to obtain the dynamics of w from the fast mode equation (3). Because of the structure of the nonlinear term, we must consider the tensor W, with elements $W_{ij} = 2\overline{\partial_i h_> \cdot \partial_j h_>}$ where the overline denotes a spatial average over the coarse-graining scale L. It is convenient to introduce the curvature tensor K, with elements $K_{ij} = 2\partial_i \partial_j h_<$. In this notation, $w = \frac{1}{2}\mathrm{Tr}\mathsf{W}$ and $\kappa \equiv \nabla^2 h_< = \frac{1}{2}\mathrm{Tr}\mathsf{K}$. Taking the time derivative of W_{ij} and using (3), we obtain

$$\partial_t \mathsf{W} = \mathsf{F}[\mathsf{W}] + \mathsf{W} \cdot \mathsf{K} + \mathsf{K} \cdot \mathsf{W}, \tag{5}$$

where $\mathsf{F}[\mathsf{W}]$ contains purely fast mode dynamics and will be described shortly. The forms of the last two terms in Eq. (5) are fixed by the Galilean invariance of the KS equation and are exact.

Equation (5) can be made more transparent by rewriting the two tensors as $\mathsf{W} = w \cdot \mathsf{1} + \tilde{w} \cdot \mathsf{Q}(\phi)$ and $\mathsf{K} = \kappa \cdot \mathsf{1} + \tilde{\kappa} \cdot \mathsf{Q}(\theta)$, where $\mathsf{1}$ is the identity matrix and $\mathsf{Q}(\alpha)$ is a unit *traceless* matrix, represented by an angle α, e.g., $Q_{12}(\alpha) = Q_{21}(\alpha) = \sin(2\alpha)$ and $Q_{11}(\alpha) = -Q_{22}(\alpha) = \cos(2\alpha)$. Adopting vector notation $\vec{\psi} = (\tilde{w}\cos 2\phi, \tilde{w}\sin 2\phi)$ and $\vec{\chi} = (\tilde{\kappa}\cos 2\theta, \tilde{\kappa}\sin 2\theta)$, Eq. (5) can be rewritten as

$$\partial_t w = f[w] + 2\kappa w + 2\vec{\chi} \cdot \vec{\psi}, \tag{6}$$

$$\partial_t \vec{\psi} = \vec{\varphi}[\vec{\psi}] + \kappa\vec{\psi} + w\vec{\chi}, \tag{7}$$

to leading order, with f and $\vec{\varphi}$ obtained from the appropriate decomposition of F.

Equations (6) and (7), together with the slow mode equation (4), form a closed set of coarse-grained equations which specifies the dynamics of $h_<$ once the effective forms of the small scale dynamics, i.e., f and $\vec{\varphi}$, are given. These equations are constructed from symmetry considerations, and can be regarded as the more complete generalization of Yakhot's conjecture for two dimensions. We first discuss the physical meaning of the coarse-grained variables appearing in W and K.

The tensor K describes the local curvature of the slow modes $h_<$. With $\tilde{\kappa} = 0$, we have a symmetric paraboloid—a "valley" if $\kappa > 0$ or a "hill" if $\kappa < 0$. With $\kappa = 0$, we have a "saddle," with $\tilde{\kappa}$ and θ specifying the strength and the orientation, respectively. The tensor W characterizes the local *packing* of the cells. As in the 1D case [6], w gives the local cell density. The traceless component of W describes the local *anisotropy* in cell packing. Figure 1(a) shows an example of an arrangement of cells where $\vec{\psi}$ is pointed in the y direction. Fluctuations in the anisotropic part of the curvature K will affect the local cell packing. For example, the cell density at the bottom of a valley will be higher and a saddle configuration in $h_<$ will induce anisotropy. Equations (6) and (7) describe these effects of curvature quantitatively, much like the relation between stress and strain in elastic systems. Cell packing in turn influences the slow mode dynamics via the w term in Eq. (4). The anisotropic parts of K and W are invariant upon a rotation by 180° [see Fig. 1(a)]. Thus, we can view the vector field $\vec{\psi}(\mathbf{r}, t)$ as a "nematic" order parameter describing the local cellular orientation, and $\vec{\chi}$ as an applied field biasing $\vec{\psi}$ towards a specific orientation.

If we turn off the applied field κ and $\vec{\chi}$ in Eqs. (6) and (7), we have $\partial_t w = f[w]$ and $\partial_t \vec{\psi} = \vec{\varphi}[\vec{\psi}]$ for each coarse-grained region; thus f and $\vec{\varphi}$ describe the small scale dynamics. Even for a coarse-grained region of a few ℓ_0's, the small scale dynamics of h are already *chaotic*. The fields $w(t)$ and $\vec{\psi}(t)$ are "projections" of this small scale chaotic dynamics. They can be quantitatively characterized numerically, as we will present shortly. Before doing so, we first construct some possible scenarios.

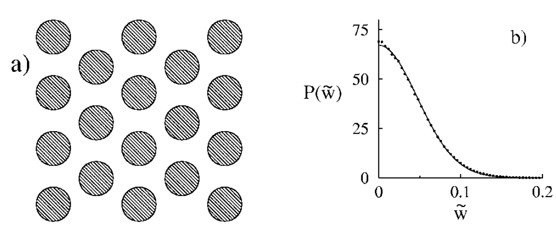

FIG. 1. (a) Example of a cellular arrangement where $\vec{\psi}$ is nonzero with orientation $2\phi = 90°$, reflecting a larger "compression" in the y direction. (b) Probability density of $\tilde{w} = |\vec{\psi}|$. The solid line is a Gaussian fit.

We expect that the h field has on average a finite drift rate, i.e., a finite time-averaged value of w. The simplest dynamics of w is then $\partial_t w(\mathbf{r}, t) = f[w] = -\alpha(w - w_0) + \xi(\mathbf{r}, t)$, where ξ is a stochastic forcing that mimics the chaotic small scale dynamics, and w_0 is a constant. This yields $w(\mathbf{r}, t \to \infty) \to w_0$. The behavior of $\vec{\psi}$ is less straightforward. In the simplest scenario, we have $\partial_t \vec{\psi} = -\tilde{\alpha}\vec{\psi} + \vec{\zeta}(\mathbf{r}, t)$ to leading order, with $\vec{\zeta}$ being a vectorial stochastic forcing. Equation (7) then yields (in the hydrodynamic limit) $\vec{\psi} \simeq (w_0/\tilde{\alpha})\vec{\chi}$, where we took the asymptotic result $w = w_0$ and assumed that the typical curvature κ is small. Note that in this scenario, the cellular orientation *passively* follows the curvature. In particular, there is no orientational anisotropy on average if there is no external forcing. Inserting this result and $f[w]$ into (6), we find in the hydrodynamic limit

$$w \simeq w_0 + \frac{2w_0}{\alpha} \nabla^2 h_< + \frac{\xi}{\alpha} + O((\partial_i \partial_j h_<)^2). \quad (8)$$

Substituting (8) into (4) yields an equation for $h_<$ of the KPZ form (2) to leading order, with $\nu = (2w_0/\alpha) - 1$ and $\eta(\mathbf{r}, t) = \xi(\mathbf{r}, t)/\alpha$. Dynamics of the KPZ universality class will be obtained if $\nu > 0$ and the noise ξ is uncorrelated between different coarse-graining regions.

Unlike the constant α however, there is no *a priori* reason why the constant $\tilde{\alpha}$ cannot be negative. This would be the case if the microscopic chaotic dynamics has a preference for the *spontaneous* breaking of local isotropy. If $\tilde{\alpha} \le 0$, then the dynamics of $\vec{\psi}$ would be more complicated. Higher order terms, e.g., $|\vec{\psi}|^2\vec{\psi}$, will be needed for stability. The minimal equation for (7) becomes

$$\partial_t \vec{\psi} = -\tilde{\alpha}\vec{\psi} - \tilde{\beta}|\vec{\psi}|^2\vec{\psi} + \gamma\nabla^2\vec{\psi} + w_0\vec{\chi} + \vec{\zeta}(\mathbf{r}, t), \quad (9)$$

where $\tilde{\beta}$ is a positive constant, and the γ term describes the coupling of neighboring coarse-grained regions. Equation (9) describes the relaxational dynamics of a nematic liquid crystal under an applied "field" $\vec{\chi}$. Its behavior depends crucially on the dynamics of the phase field ϕ, which is the Goldstone mode associated with symmetry breaking. The latter in turn depends on the parameters of Eq. (9), particularly the coupling constant γ and the amplitude of the noise $\vec{\zeta}$. The possibilities along with the effects on $h_<$ are as follows.

(i) If the noise $\vec{\zeta}$ dominates over the spatial coupling γ, then the local anisotropy will be destroyed at large scales due to the proliferation of topological defects (disclinations) in ϕ. Isotropy is restored and the KPZ universality class is recovered.

(ii) If the spatial coupling is large, then the direction of $\vec{\psi}$ may "phase lock" with the direction of $\vec{\chi}$, as manifested by $\langle(\theta - \phi)^2\rangle \ll 1$. Solving for the steady state of w in this case gives $w \simeq w_0 + \frac{w_0}{\alpha}\kappa + \frac{\tilde{w}_0}{\alpha}\tilde{\kappa}$, leading to a slow mode equation which is explicitly *not* KPZ-like since $\tilde{\kappa} = [(h_{xx} - h_{yy})^2 + 4h_{xy}]^{1/2}$. [In the KPZ case, $\tilde{\kappa}$ comes in at second order and is presumed irrelevant; see Eq. (8).]

(iii) For intermediate parameters, there may exist a "spin wave" phase characterized by $\langle\phi(\mathbf{r})\phi(0)\rangle = a\log|\mathbf{r}|$. Here, spin wave fluctuations would add a long range component to the effective KPZ noise, since $\langle\cos[\phi(r) - \phi(0)]\rangle \sim r^{-a}$. For sufficiently small a, it would yield dynamics that are not in the KPZ universality class.

To distinguish between these scenarios, we numerically measured the "fast mode" dynamics with simulations on systems of size $L \times L$, with L ranging from 32 to 128 with periodic boundary conditions. (A single cell had a length $\ell_0 \approx 8.9$.) We used a simple spatial discretization scheme with a Euler time step of 0.02. We first checked for spontaneous symmetry breaking by examining the distribution of $\vec{\psi}$. We measured $\vec{\psi}$ spatially averaged over a system of size $L = 32$, sampling at time intervals of $t = 4$ over a total period of $t = 4.8 \times 10^5$. The distribution of $|\vec{\psi}| = \tilde{w}$ (normalized by the cylindrical area) is shown in Fig. 1(b). The Gaussian shape (within ~3%) indicates a lack of spontaneous symmetry breaking, strongly supporting the KPZ scenario.

We next measured the response of W to an imposed curvature tensor K using methods similar to those described in Ref. [6]. In an STC steady state, we abruptly turned on a forcing term in Eq. (3) of the form $h_<(x, y) = ck^{-2}\sin k_x x \sin k_y y$, with $k_x = k_y = \pi/2L$, for a range of amplitudes c. This configuration yielded eight separate overlapping regions of size $L/2 \times L/2$: four hill and valley regions, and four saddle regions with orientations $\theta = \pi/4$ and $\theta = 3\pi/4$. To account for the spatial dependence of the curvature K within each forcing region, we simply took the imposed curvature to be the average curvature. This gave $\kappa = \overline{\nabla^2 h_<} = (4/\pi^2)c$. We zeroed the first three Fourier modes of $h_>$ after each time step to remove the nonlinear slow mode response to the forcing (see Ref. [6]).

We found that w responded only to κ and $\vec{\psi}$ responded only to $\vec{\chi}$, further validating the KPZ scenario. Figure 2(a) shows an example of the time dependent response of the drift rate w to the forcing. The saturated amplitude $A_L(c)$

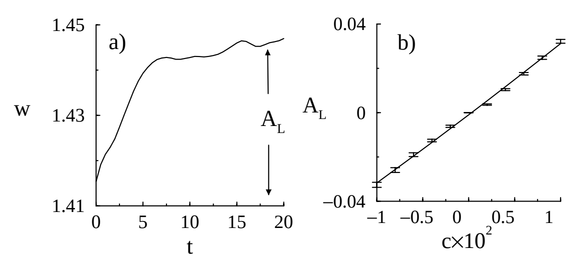

FIG. 2. (a) Response of the drift velocity w to forcing with $c = 0.1$ applied at $t = 0$ for $L = 128$. (b) Saturated amplitude $A_L(c)$; the slope is 3.15 ± 0.06.

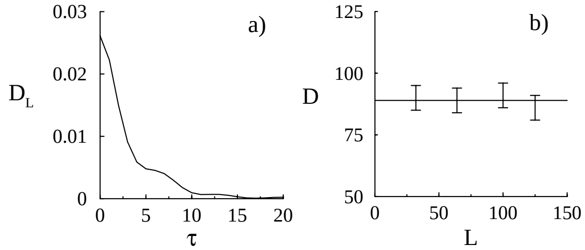

FIG. 3. (a) Correlator for $L = 32$; (b) noise amplitude for different L's.

was a linear function of c, as seen in Fig. 2(b). In accordance with Eq. (8), the ratio $2w_0/\alpha$ was computed from the slope, from which we obtained $\nu = 14.9 \pm 0.5$ for $L = 64$ and $\nu = 14.5 \pm 0.5$ for $L = 128$. The averaged response times were $\alpha^{-1} = 3.3 \pm 0.2$ for $L = 64$ and $\alpha^{-1} = 4 \pm 0.2$ for $L = 128$. Similar behaviors were obtained in the hyperbolic forcing region. We verified that $\vec{\psi} \propto \vec{\chi}$, and found that $\tilde{\alpha}^{-1} = 2.0 \pm 0.1$ for $L = 64$ and $\tilde{\alpha}^{-1} = 3.0 \pm 0.3$ for $L = 128$.

To characterize the effective stochasticity, we measured the two-point correlation function $C_L(\tau) = \langle [w(t + \tau) - w(t)]^2 \rangle$ of the drift rate w for systems of sizes ranging from $L = 32$ to $L = 128$. The relevant quantity is the correlator $D_L(\tau) = [C_L(\infty) - C_L(\tau)]/2$ [see Fig. 3(a)]. For short range correlated noise, the effective noise amplitude $D = L^2 \int_0^\infty D_L(\tau)\, d\tau$ is expected to be independent of L. Figure 3(b) shows measurements of D, with the average $D = 89 \pm 5$. The numerical values of the effective parameters D and ν extracted using our coarse-graining scheme are in reasonable agreement with that of Ref. [12].

In summary, we have performed an analysis of the 2D-KS equation at the mesoscopic scale of several cell sizes. By using an explicit coarse-graining scheme, we constructed various possibilities for the effective equation of motion for the slow modes $h_<$. We found, as pointed out before in [10,11], that the KPZ universality class is not the only possibility for the 2D-KS equation. Since our analysis is confined to the mesoscopic (large-k) limit, the various scenarios obtained are all nonperturbative and nonlocal in k. The more interesting scenarios involve spontaneous breaking of rotational symmetry. Although we found no symmetry breaking for the KS equation numerically, our analysis indicates that such solutions are allowed by symmetry, and may occur in other KS-like systems. For the KS equation proper, we conclude that it belongs to the KPZ universality class. However, for all practical purposes, the behavior is well described by a stochastic diffusion equation with logarithmic corrections.

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[13] Their argument is contingent upon an equality [Eq. (23) of Ref. [10]] relating the difference of two integration constants (called C_1 and C_2) and the bare coefficient of the diffusion term. The nonlocal solution is tenable if the equality is satisfied. They claim that the constants, which come from Wyld diagrammatic calculations, are determined *uniquely* by the $(\nabla h)^2$ nonlinearity and are therefore the *same* for both the KS and the KPZ equations. However, the calculation involved integration of the "full" response and correlation functions over the *entire* range of k. We note that these functions should be different for large values of k's where the microscopic dynamics matter. Consequently the constants C_1 and C_2 need *not* be the same for the KS and KPZ equations.
[14] The nonlocal solution of Ref. [10] has the same form as that of the $g = 0$ fixed point of the 2D-KPZ equation. For the latter, multiplicative logarithmic correction to diffusion arises from the "asymptotic freedom" of the system as $g \rightarrow 0^-$. The conjectured exponents of the logarithmic correction factor follow naturally from the nonrenormalization of ν of the KPZ equation to all orders in λ.
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