

Magnetic-Field-Induced Directional Localization in a 2D Rectangular Lattice

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We study the effect of a perpendicular uniform magnetic field on the dissipative conductivity of a rectangular lattice with anisotropic hopping, $t_x \neq t_y$. We show that the magnetic field may enhance dramatically the directional anisotropy in the conductivity. The effect is a measurable physical realization of Aubry's duality in Harper systems.

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Localization in aperiodic systems has been at the center of attention for decades in condensed matter physics. Most work concerned disordered systems. Twenty years ago, however, Aubry *et al.* [1] predicted that a 1D tight-binding Hamiltonian with a quasiperiodic potential exhibits a metal-insulator transition as the amplitude of the potential becomes larger than a critical value. The proof rests on a duality property that allows mapping low into large coupling constants, while corresponding extended wave functions are transformed into states that are localized. The tight-binding model used by Aubry leads to the almost Mathieu equation [2,3], a special case of a more general class of quasiperiodic systems for which the duality applies. It also happens that the almost Mathieu equation arises in the study of the dynamics of electrons in 2D in the presence of a rectangular lattice and a perpendicular uniform magnetic field (see [4] for the case of a square lattice). In this case Aubry's duality may be interpreted as a rotation by $\pi/2$ of the lattice, an operation that can be performed easily in a real sample and thus lends itself to experimental test. The aim of this Letter is to show that due to Aubry's duality, turning on a magnetic field may produce a dramatic enhancement of the anisotropy already present in the conductivity of a rectangular potential. We denote by t_x, t_y the hopping amplitudes along the x and the y axes (with $t_x < t_y$), and by σ_{xx}, σ_{yy} the corresponding longitudinal conductivities. At zero magnetic field the Drude formula away from the parabolic edges of a tight-binding band yields $(\sigma_{xx}/\sigma_{yy})_0 \approx \text{const}(t_x/t_y)^2$ for small t_x/t_y . With magnetic field, however, we obtain within the relaxation time approximation

$$\left. \frac{\sigma_{xx}}{\sigma_{yy}} \right|_B \approx \frac{\gamma}{2} \left(\frac{\hbar}{\tau} \pi n(\mu) \right)^2 \left. \frac{\sigma_{xx}}{\sigma_{yy}} \right|_0, \quad (1)$$

for irrational flux per plaquette. Here $n(\mu)$ is the density of states per unit cell at the Fermi energy μ , τ is the dominant scattering time, and γ is a constant of $O(1)$. In deriving this result we assume that the temperature is low enough so that $kT < \hbar/\tau$ and that the Fermi level is not too close from the edge of a gap larger than or equal to $O(k_B T, \hbar/\tau)$. It may fail as well in a regime where

Mott's variable range hopping dominates. The enhanced asymmetry exhibited by Eq. (1) for a large relaxation time is physically understandable in terms of Aubry's duality: the electronic eigenstates are extended along the easy direction y leading to a metalliclike behavior for σ_{yy} , whereas they are localized in the direction x , leading to an insulatinglike behavior for σ_{xx} . We shall argue that the predicted enhancement should be observable in a superlattice of quantum dots.

Consider a tight-binding Hamiltonian in the x - y plane. For convenience we choose the gauge $\vec{A}_x = B(0, x, 0)$ in which the coordinate y becomes cyclic, permitting plane wave solutions along this latter axis. The wave function along x must then obey the almost Mathieu difference equation [5]

$$2t_y \cos(2\pi\alpha m - k_y b)\psi(ma) + t_x[\psi((m+1)a) + \psi((m-1)a)] = E\psi(ma). \quad (2)$$

Here the field variable $\alpha = eBab/hc$ gives the number of flux quanta traversing the unit cell, E is the energy, a and b are the lattice constants in the x and the y direction, respectively, k_y is the wave number of the free running plane wave along the y axis, and m is an integer labeling the lattice sites. The conventional Harper model [4] is obtained by making the lattice square, with $a = b$ and $t_x = t_y$. We are interested in the asymmetric case, usually arising from the unit cell being rectangular, although a square array of elliptical quantum dots, for example, would also provide the required asymmetry.

Inspection of Eq. (2) shows that in the limit $t_x/t_y \gg 1$, the solutions are plane waves, slightly modulated by the quasiperiodic potential. In the other extreme $t_x/t_y \ll 1$ however, the solutions are localized features a distance qa apart if $\alpha = p/q$ is rational, or a single localized feature if this parameter is irrational [6]. For t_x/t_y finite one can show that under Fourier transformation Eq. (2) formally turns into itself, with the roles of t_x and t_y exchanged. An extended state obtained for $t_x/t_y \gg 1$ is thus replaced by a localized state in Fourier transformed space. This property is known as Aubry's duality [1]. Another way of obtaining an exchange of roles of t_x and t_y is to

change the gauge. To see this assume that $t_x/t_y \ll 1$ so that the states given by Eq. (2) are localized along the x axis. If in the original problem one uses the gauge $A_y = B(-y, 0, 0)$ instead of $A_x = B(0, x, 0)$, the resulting equation is formally identical to Eq. (2), only that now the wave function describes the dynamics along y , and t_x, t_y exchange places. Because of this latter fact and our assumption about the relative size of these parameters the new version of Eq. (1) gives now extended states that run along the y direction. Thus, while one gauge yields states that are localized along the hard hopping direction, the other gauge yields extended states along the easy hopping direction. This means that Aubry's duality is manifested in a single sample, its two principal axes playing the role of the dual states. As we show below, the anisotropy in the conductivity may reveal this effect in a dramatic way.

The spectrum of Eq. (2) for irrational values of the field parameter is a Cantor set with a hierarchy of gaps that become extremely small [2,3,6]. Figure 1 shows the spectrum for different values of t_x/t_y . Note that as this ratio decreases from 0.8 to 0.2 the spectral gaps become more and more invisible due to their diminishing size. Each gap is uniquely labeled by an integer s [7] taking values between $-(q-1)/2$ and $(q-1)/2$ if $\alpha = p/q$,

and all values if α is irrational. It has been shown that the gap width behaves like [3]

$$\Delta_s \sim (t_x/t_y)^{|s|} t_y, \quad (3)$$

for small t_x/t_y . For a rational $\alpha = p/q$, the spectrum has exactly q subbands that do not overlap so that up to $q-1$ gaps may appear. These are actually all open except for the one at $E=0$ for q even [3,8]. For irrational α the spectrum can be well approximated by the rational approximants p_n/q_n [9] obtained from truncating the continuous fraction expansion of α at its n th step [10]. The gap labels are then stable through this approximation [7,9]. In real samples the presence of scattering limits the experimental access to such a fine structure [11]. There is thermal broadening of size $k_B T$, and also an energy width \hbar/τ associated with other sources of scattering, with τ a characteristic relaxation time. All measurable quantities will be rather insensitive to gaps smaller than $\delta = \max(k_B T, \hbar/\tau)$. Moreover, only energies within an interval of size δ from the Fermi energy μ will matter for the electronic transport. Therefore, if α is irrational, it is sufficient to replace it by its best rational approximant p/q such that the gaps closest to μ have width not smaller than δ . The error introduced in transport properties by

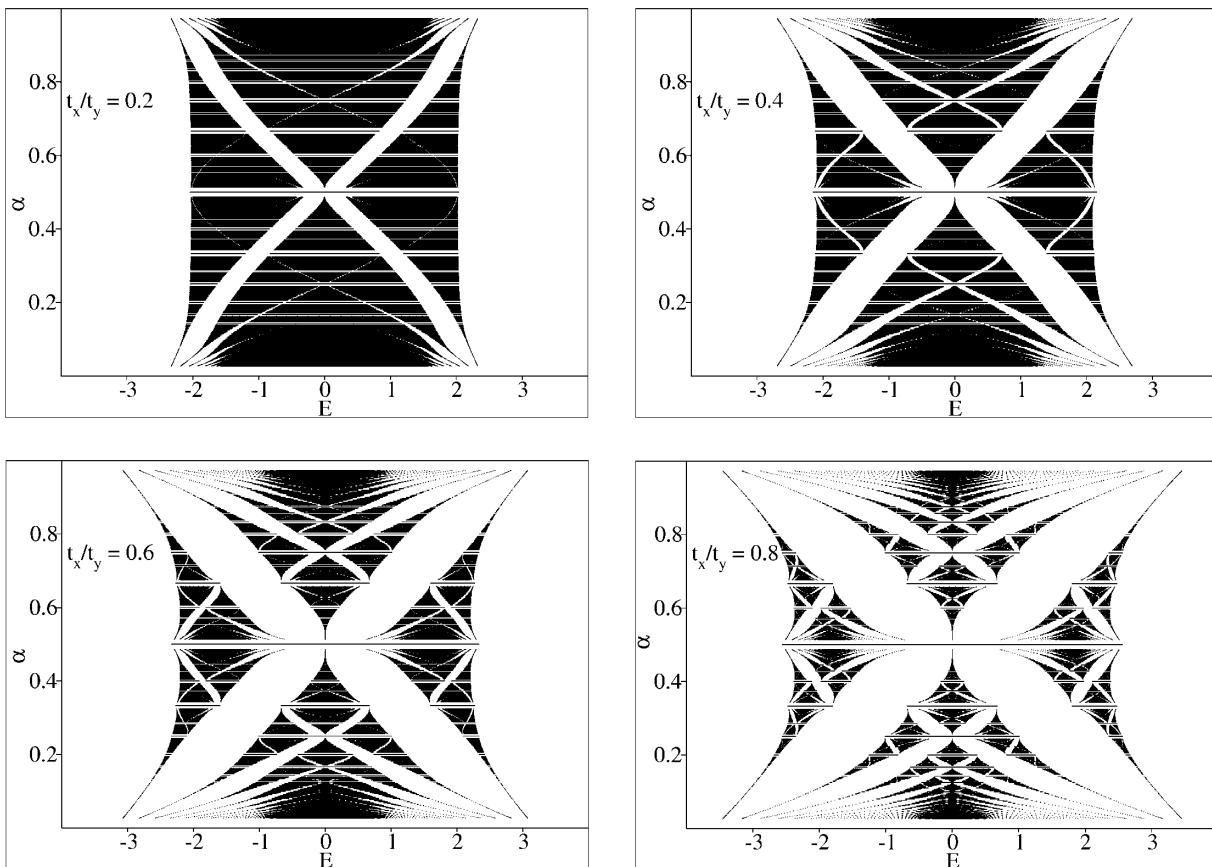


FIG. 1. Spectra of the Harper problem on a rectangular lattice for $t_x/t_y = 0.8, 0.6, 0.4$, and 0.2 obtained for rational values of $\alpha = p/q$ with $q \leq 37$.

this substitution is of the order of the hopping probability between sites at distance qa apart, which for t_x small is bounded to be also very small. In our calculation of the conductivity we shall adopt this simplifying criterion.

More precisely, for each integer n let $\Delta^{(n)}$ be the minimal width of the closest gaps to μ corresponding to the rational approximant p_n/q_n , and let s_n be the corresponding gap label. Then one chooses the largest value N of n such that

$$\Delta^{(N+1)} < \delta = \max\left(k_B T, \frac{\hbar}{\tau}\right) \leq \Delta^{(N)}, \quad (4)$$

thus fixing the values of $q = q_N$ and $s = s_N$. In particular, as the temperature T decreases, one expects τ to increase, so that the value of N increases as well and with it, those of q_N and s_N .

A convenient form of Eq. (2) for the rational case is

$$2t_y \cos\left(2\pi m \frac{p}{q} - k_y b\right) \phi_\ell(m) + t_x \times [e^{ik_x a} \phi_\ell(m+1) + e^{-ik_x a} \phi_\ell(m-1)] = E_\ell(k_x, k_y) \phi_\ell(m), \quad (5)$$

where m is an integer, $\phi_\ell(m)$ is periodic of period q , and k_x is a phase. For each point $k = (k_x, k_y)$ in phase space, with $0 \leq k_x a, k_y b \leq 2\pi$, there are q eigenvalues which, as \vec{k} covers its range, make up the q subbands labeled by the integers $\ell = 1, 2, \dots, q$. All eigenvalues are contained in the energy interval $|E_\ell| < 2(t_x + t_y) = W/2$, with W the width of the original zero-field band. We assume the Fermi energy μ to lie somewhere within this range.

In the infinite volume limit and in the relaxation time approximation, the longitudinal conductivity for our q subbands system is given by Kubo's formula [12]

$$\sigma_{ii} = \frac{2e^2 \tau}{q \hbar^2} \sum_{\ell=1}^q \int \frac{d^2 k}{4\pi^2} \left| \frac{\partial E_\ell}{\partial k_i} \right|^2 \delta(E_\ell - \mu) + \frac{4e^2}{q \hbar^2} \times \text{Re} \sum_{\substack{\ell \neq \ell'=1 \\ E_{\ell'} < \mu < E_\ell}}^q \int \frac{d^2 k}{4\pi^2} \frac{|\langle \phi_{\ell'} | \partial \phi_\ell / \partial k_i \rangle|^2 (E_\ell - E_{\ell'})}{1/\tau - i(E_\ell - E_{\ell'})/\hbar}, \quad (6)$$

where E_ℓ and $E_{\ell'}$ depend upon the phase space coordinates (k_x, k_y) and the integrals are taken over all of phase space. Following condition (4), in Eq. (6) we have replaced the Fermi distribution by a step function. The first term in this equation is the intraband term, while the second includes all interband contributions. In the absence of a magnetic field there is just one band in our model and only the first term is relevant. In what follows we shall assume $t_x < t_y$.

A careful analysis of Eq. (6) shows that the conductivity depends strongly on the position of the Fermi energy with respect to subband edges. We distinguish two cases.

(1) μ far from the subband edges.—Assume first that the Fermi energy is at a distance larger than $O(\Delta^2/t_y)$ from the nearest gap. The intraband term can be computed solving the eigenvalue equation

$$P_q(E) = 2(-)^{q+1} (t_x^q \cos q a k_x + t_y^q \cos q b k_y), \quad (7)$$

for the appropriate subband ℓ . In this expression P_q is the Chambers polynomial associated with Eq. (5) [3,13]. Because of condition (4), the interband contribution in Eq. (6) can be expanded in $1/\tau$ since $\hbar/\tau < |E_\ell - E_{\ell'}|$. Keeping just the first terms in this expansion, one gets to lowest order in $\lambda = t_x/t_y$

$$\sigma_{xx} = \left(\frac{e}{\pi \hbar}\right)^2 \frac{2r\tau}{n(\mu)} \frac{\lambda^{2q}}{(1-\xi^2)} + \gamma \frac{e^2 r n(\mu)}{\tau} \lambda^2 \quad (8)$$

$$\sigma_{yy} = \left(\frac{2e}{\pi \hbar}\right)^2 \frac{\tau}{r n(\mu)}. \quad (9)$$

Here, as mentioned earlier, $n(\mu)$ is the density of states at the Fermi energy and γ is a numerical factor of order 1, while $r = a/b$ and $\xi = P_q(\mu)/2t_y^q$. The first term in these expressions is the Drude conductivity coming from the intraband transitions in Eq. (6). That the ratio $(\sigma_{xx}/\sigma_{yy})_{\text{Drude}}$ is of $O(\lambda^{2q})$ is apparent from the energy derivative in Eq. (6) and the fact that the Chambers polynomial depends on the phases through the constant term in the right hand side of Eq. (7), only. Also, that the lowest order contribution to the interband transitions is $O(\lambda^2)$ follows from the fact that in a perturbative expansion in terms of λ , the zeroth order term in ϕ_ℓ does not depend on the phases.

Assuming $q_N > 1$ and having in mind that at sufficiently low temperatures and within the rational approximation ansatz $|s_N| \leq (q_N - 1)/2$ and $\hbar/\tau t_y \sim \Delta^{(N)}/t_y \sim \lambda^{|s_N|} \geq \lambda^{(q_N-1)/2}$, one finds the ratio between the Drude and interband contributions to the conductivity along the x direction to be negligible. Ignoring the first term in (8) we then have

$$\frac{\sigma_{xx}}{\sigma_{yy}} \approx \gamma \left(\frac{\hbar}{\tau} \frac{\pi r n(\mu)}{2} \lambda \right)^2. \quad (10)$$

This important result states that as long as μ is not too close to a subband edge of the proper rational approximant, the conductivity ratio vanishes quadratically in the inverse relaxation time for small λ . Applying Eq. (6) to the original tight-binding band to obtain the corresponding conductivity ratio in the absence of a magnetic field near the band center one then arrives at the form given by Eq. (1).

(2) μ near a subband edge.—When the Fermi energy μ is very near a subband edge the situation changes significantly. The density of states has a logarithmic singularity in that neighborhood, and $n(\mu) \propto (t_y/t_x)^{q_n/2}$

in a region of order Δ^2/t_y from the edges. As a result, Eqs. (8) and (9) are no longer correct. The expressions for the Drude contribution are valid in the region $0 < |\xi| < 1 - \lambda^{q_N}$ only. In the range $1 - \lambda^{q_N} < |\xi| \leq 1 + \lambda^{q_N}$ or if the Fermi energy lies in a gap, σ_{xx} may become as large as σ_{yy} . A detailed discussion of this regime will be published elsewhere [14]. These situations may not be relevant for experiments however, because they occur only in a very small region of the spectrum. Indeed, when $t_x < t_y$, the probability of having μ within $O(\Delta^2 < t_y)$ from the subband edge behaves like $(t_x/t_y)^2$. In addition, for q_N large the total length of the subbands is given by $4|t_y - t_x|$ [1,15,16]. Thus, choosing the Fermi energy μ at random in the energy interval $[-W/2, W/2]$ gives μ in one subband with relative probability

$$\frac{|t_y - t_x|}{t_x + t_y} \sim 1 - 2 \frac{t_x}{t_y}. \quad (11)$$

Thus, the probability of having μ in a gap or close to a gap edge is proportional to t_x/t_y which is small [14]. Moreover, it is likely that very small fluctuations in the magnetic field will tend to wash away the effects of μ lying in the anomalous regions close to a subband edge or in a gap.

In summary, we have shown that in the presence of a perpendicular magnetic field the field free conductivity asymmetry of a rectangular lattice may be dramatically enhanced in a pure sample. This is a physical realization of Aubry's duality that may be understood in the following way. The magnetic field affects the hopping of the tight-binding electrons through the effective potential of period q times the appropriate lattice constant. The site energies are then no longer the same within and the electron has to tunnel a distance equal to this period, through a potential mismatch that scales with λ^{-1} or λ , depending on whether transport is along the x or y direction. For an irrational field the tunneling can be made possible only through scattering events, which are less likely as the relaxation time diverges. Evidence for this behavior has also been found in the spreading of a wave packet in the presence of a weak modulation potential in two dimensions. In the regime in which the single band approximation holds, such spreading is entirely directional, occurring along the direction of largest hopping amplitude only [17]. Our results also show that the suppression of the conductivity along one of the principal axes by an external field is controlled by the ratio of the gap size to the zero-field bandwidth $(\hbar/\tau)(1/W) \sim (t_x/t_y)^{|s_N|}$. With present day rectangular arrays of antidots this quantity is of order one tenth. Thus the effect should already be visible. It is likely that, as the miniaturization of mesoscopic technology progresses, this quantity can be made even smaller so

that the magnetic-field-induced localization may be more easily observed.

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