Turbulent Fluctuation and Transport of Passive Scalars by Random Wave Fields

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Turbulent transport of passive scalars by random wave fields is studied, with applications to statistics of chlorophyll concentration in the ocean. The existence of the small parameter u_0/c_0, where u_0 and c_0 are the characteristic particle velocity and wave phase speed, respectively, allows essentially exact calculations, and as such provides a rich testing ground for quantitative comparisons between theory and observation. General expressions are derived for the diffusion constant and mean drift velocity. It is shown that the spectrum of passive scalar fluctuations displays at least two distinct inertial range power laws even when the wave velocity field has only one.

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Passive scalar transport by turbulent velocity fields has been a subject of intense interest for many years [1–4]. The problem is of great importance in ocean and atmosphere dynamics where the transport of heat, moisture, salt, and biogeochemical quantities has short term (weather) as well as long term (climate) implications. Theories to date have focused mainly on the effects of Navier-Stokes turbulence in two and three dimensions. The velocity field is then strongly nonlinear, and analytic solutions are restricted to approximate closure schemes, or special model problems with Gaussian statistics [4].

In this work we study transport by *traveling wave fields* [3]. It transpires that a small parameter, u_0/c_0, in the problem, where u_0 and c_0 are the characteristic particle and wave phase speed, respectively, allows essentially exact analytic treatment via direct computation of the first few orders in a straightforward perturbation theory. Closure schemes, e.g., are unnecessary since the additional higher order terms they include are negligibly small. The amazing variety of physical wave systems provide a rich set of problems that are amenable to quantitative analysis, a rare commodity in fluid dynamics. Satellite observations of different ocean surface regions then provide unique laboratories where the theory can be tested.

Below we outline the theoretical formalism, based on expanding the Lagrangian in terms of Eulerian dynamics, exploiting the smallness of u_0/c_0. We use it to derive an effective diffusion equation for the *mean* concentration field, and also evaluate the *spectrum* of fluctuations about the mean. We find that the inertial range exhibits *two* distinct power-law regions even when the velocity field has only one. The predicted spectra compare favorably with ocean chlorophyll spectra in ocean regions whose dynamics are dominated by wave motions.

The equation of motion for the passive scalar concentration field $\psi(\mathbf{x},t)$ by a (possibly compressible—e.g., acoustic waves) advecting velocity field $\mathbf{v}(\mathbf{x},t)$ is

$$\partial_t\psi + \nabla\cdot(\mathbf{v}\psi) = \kappa\nabla^2\psi\,, \qquad (1)$$

with microscopic (molecular) diffusion constant κ. Large-scale transport induced by $\mathbf{v}$ is generally many orders of magnitude greater than that by κ, and we henceforth set $\kappa \equiv 0$. For incompressible $\mathbf{v}$ the nonlinear term simplifies to $\mathbf{v}\cdot\nabla\psi$. Mean quantities are defined via an ensemble average over $\mathbf{v}$. For Navier-Stokes turbulence these statistics are strongly non-Gaussian and poorly understood, but for waves, which have a well defined set of at most weakly interacting modes, one has the representation

$$\mathbf{v}(\mathbf{x},t) = \int \frac{d^{\hat d}k}{(2\pi)^{\hat d}}\, a(\mathbf{k})\hat{\mathbf{e}}(\mathbf{k};\mathbf{z})e^{i[\mathbf{k}\cdot\mathbf{r}-\omega(\mathbf{k})t]} + \text{c.c.}\,, \qquad (2)$$

and Gaussian statistics, fully characterized by the *amplitude spectrum*

$$\langle a(\mathbf{k})a^*(\mathbf{k}')\rangle = f(\mathbf{k})\,(2\pi)^{\hat d}\delta(\mathbf{k}-\mathbf{k}')\,, \qquad (3)$$

are an excellent approximation. Here c.c. stands for complex conjugate, $\mathbf{k}$ is the wave vector, $a(\mathbf{k})$ is the mode amplitude, $\hat{\mathbf{e}}(\mathbf{k};\mathbf{z})$ is the mode profile, and $\omega(\mathbf{k})$ is the dispersion relation [e.g., $\omega(\mathbf{k}) = c|\mathbf{k}|$ for acoustic waves]. We have divided the full d-dimensional space (kept general for convenience) of $\mathbf{x} = (\mathbf{r},\mathbf{z})$ into a $\hat d \le d$ dimensional "horizontal" subspace $\mathbf{r}$, and a $\bar d = d - \hat d$ dimensional "vertical" subspace $\mathbf{z}$. For typical oceanographic applications, $\hat d = 2$ and $\bar d = 1$. The form (2) shows that the full wave-number–frequency velocity spectrum is confined to the hypersurfaces $\omega = \pm\omega(\mathbf{k})$. The spectrum $f(\mathbf{k})$ is typically peaked about some characteristic wave number k_0, which then defines a characteristic wavelength $\lambda_0 = 2\pi/k_0$ and wave period $\tau_0 = 2\pi/\omega(k_0)$. The *width* Δk of the spectrum yields a corresponding frequency width $\Delta\omega \simeq c_0\Delta k$, which define a correlation length $\xi = 2\pi/\Delta k$ and a decorrelation time $\tau = 2\pi/\Delta\omega$ of the wave field. Typically τ,ξ are a few times τ_0,λ_0. A crucial characteristic of waves is that $f(\mathbf{k}) \equiv 0$ in some finite region about $\mathbf{k} = 0$: waves of very large wavelength and/or low frequency are never physically excited.

We make use of the following random walk representation for $\psi(\mathbf{x},t)$ [5]. Let $\mathbf{Z}_{\mathbf{x}s}(t)$ be the Lagrangian trajectory of a freely advected particle, constrained to be at $\mathbf{x}$ at time s: $\partial_t \mathbf{Z}_{\mathbf{x}s}(t) = \mathbf{v}(\mathbf{Z}_{\mathbf{x}s}(t),t)$ with $\mathbf{Z}_{\mathbf{x}s}(s) = \mathbf{x}$. The equivalent integral form is

$$\mathbf{Z}_{\mathbf{x}s}(t) = \mathbf{x} + \int_s^t ds'\, \mathbf{v}(\mathbf{Z}_{\mathbf{x}s}(s'),s'). \tag{4}$$

A formal solution to (1) (with $\kappa = 0$) is then

$$\psi(\mathbf{x},t) = \int d^d x'\, \psi(\mathbf{x}',s)\delta[x - \mathbf{Z}_{\mathbf{x}'s}(t)], \tag{5}$$

for any $t > s$. The function $\psi(\mathbf{x},t)$ is a random variable, dependent on the history of $\mathbf{Z}_{\mathbf{x}s}(t)$. Only if $t - s > \tau$ is $\mathbf{Z}_{\mathbf{x}'s}(t)$ statistically independent of $\psi(\mathbf{x}',s)$ and does the average of (5) factorize into the Markov-type form

$$\begin{aligned}\bar\psi(\mathbf{x},t) &= \int d^d x'\, P(\mathbf{x},t\,|\,\mathbf{x}',s)\bar\psi(\mathbf{x}',s),\\ P(\mathbf{x},t\,|\,\mathbf{x}',s) &\equiv \langle \delta[x - \mathbf{Z}_{\mathbf{x}'s}(t)]\rangle,\\ &= \int \frac{d^d K}{(2\pi)^d}\, e^{i\mathbf{K}\cdot(\mathbf{x}-\mathbf{x}') - \lambda(\mathbf{K};\mathbf{x}',t-s)},\end{aligned} \tag{6}$$

where $\bar\psi \equiv \langle\psi\rangle$. In the last line the usual Fourier representation of the δ function has been used, $\mathbf{K}$ is a full d-dimensional wave vector, $\lambda = -\ln\langle e^{-i\mathbf{K}\cdot\Delta\mathbf{Z}_{\mathbf{x}0}(t)}\rangle$, and $\Delta\mathbf{Z}_{\mathbf{x}0}(t) \equiv \mathbf{Z}_{\mathbf{x}0}(t) - \mathbf{x}$. A transport equation for $\bar\psi$ is derived by taking the time derivative of (6), bringing down a factor $\rho \equiv \partial_t\lambda$. Diffusion and mean drift are large-scale phenomena that emerge on length and time scales much larger than the ξ and τ. To study them one then performs a Taylor expansion of ρ for small $|\mathbf{K}|$:

$$\begin{aligned}\rho(\mathbf{K};\mathbf{x},t) &= -\sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\sum_{l_1,l_2,\ldots,l_n=1}^{d}\rho^{(n)}_{l_1 l_2\ldots l_n}(\mathbf{x},t)\\ &\quad\times K_{l_1}K_{l_2}\cdots K_{l_n},\\ \rho^{(1)}_l(\mathbf{x},t-s) &\equiv \langle v_l(\mathbf{Z}_{\mathbf{x}s}(t),t)\rangle,\\ \rho^{(2)}_{lm}(\mathbf{x},t-s) &\equiv \langle v_l(\mathbf{Z}_{\mathbf{x}s}(t),t)\Delta Z^l_{\mathbf{x}s}(t)\rangle_c + (l\leftrightarrow m),\end{aligned} \tag{7}$$

and so on. The subscript c indicates a cumulant average: $\langle v_l\rangle\langle\Delta Z^m\rangle + (l\leftrightarrow m)$ should be subtracted. The multitime Lagrangian correlators $\rho^{(n)}$, in fact, become time independent for $t - s > \tau$. Substituting (7) into (6), using the correspondence $iK_l \leftrightarrow \partial_l$, one obtains a *gradient expansion* for the equation of motion:

$$\begin{aligned}\partial_t\bar\psi(\mathbf{x},t) &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}\sum_{l_1,\ldots,l_n}\partial_{l_1}\cdots\partial_{l_n}\\ &\quad\times\int d^d x'\, P(\mathbf{x},t\,|\,\mathbf{x}',s)\bar\psi(\mathbf{x}',s)\rho^{(n)}_{l_1\ldots l_n}(\mathbf{x}').\end{aligned} \tag{8}$$

If $\rho^{(n)}(\mathbf{x})$ varies slowly on the scale of the dependence of P on $\mathbf{x} - \mathbf{x}'$, i.e., if the statistics of $\mathbf{v}$ change very slowly on the scale of ξ, one may factor $\rho^{(n)}$ out of the integral to obtain the *local* equation of motion,

$$\begin{aligned}\partial_t\bar\psi(\mathbf{x},t) &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}\sum_{l_1,\ldots,l_n}\partial_{l_1}\ldots\partial_{l_n}\\ &\quad\times[\rho^{(n)}_{l_1\ldots l_n}(\mathbf{x})\bar\psi(\mathbf{x},t)].\end{aligned} \tag{9}$$

This factorization is exact for a translation invariant system where the $\rho^{(n)}$ are $\mathbf{x}$ independent. It is approximate if $\bar d > 0$ since the $\rho^{(n)}$ will then depend on the vertical coordinate $\mathbf{z}$. For sufficiently smooth $\bar\psi$ one may drop all terms for $n \geq 3$ to obtain the diffusion equation,

$$\partial_t\bar\psi + \nabla\cdot(\mathbf{u}\bar\psi) = \nabla\cdot(\mathbf{D}\cdot\nabla\bar\psi), \tag{10}$$

with mean drift velocity and diffusion tensor,

$$\begin{aligned}u_l(\mathbf{x}) &= \rho^{(1)}_l(\mathbf{x}) - \frac{1}{2}\sum_m \partial_m\rho^{(2)}_{lm}(\mathbf{x}) + \ldots\\ D_{lm}(\mathbf{x}) &= \frac{1}{2}\rho^{(2)}_{lm}(\mathbf{x}) - \frac{1}{6}\sum_n\partial_n\rho^{(3)}_{lmn}(\mathbf{x}) + \ldots.\end{aligned} \tag{11}$$

Results (9)–(11) are general, not restricted to waves, but explicit computation of the $\rho^{(n)}$ is often impossible due to the nonlinear relation (4) between Lagrangian and Eulerian coordinates. We show now, however, that for wave fields a controlled calculation is possible.

Equation (4) may be iterated to obtain the following time-ordered product Eulerian expansion:

$$\begin{aligned}\mathbf{v}(\mathbf{Z}_{\mathbf{x}s}(t),t) &= \sum_{n=0}^{\infty}\int_s^t ds_1\int_s^{s_1} ds_2\ldots\int_s^{s_{n-1}} ds_n\\ &\quad\times[\mathbf{v}(\mathbf{x},s_n)\cdot\nabla][\mathbf{v}(\mathbf{x},s_{n-1})\cdot\nabla]\cdots\\ &\quad\times[\mathbf{v}(\mathbf{x},s_1)\cdot\nabla]\mathbf{v}(\mathbf{x},t),\end{aligned} \tag{12}$$

in which the gradient operators act on *all* $\mathbf{x}$ dependence to their right. For waves this expansion is rapidly convergent: $\mathbf{v}$ is $O(u_0)$ and varies spatially on scale λ_0, hence $\nabla\mathbf{v} = O(u_0/\lambda_0)$. For $t - s = O(\tau)$, the nth term in (12) is then $O[u_0(u_0\tau/\lambda_0)^n]$: if the distance $u_0\tau$ traveled by a tracer particle in a decorrelation time is much less than the wavelength λ_0, convergence is assured. For $\tau \sim \tau_0$ one has $\tau \sim \lambda_0/c_0$ and hence $u_0\tau/\lambda_0 \sim u_0/c_0$. Since u_0 is proportional to the wave amplitude, while c_0 is independent of it, one will have $u_0/c_0 \ll 1$ for low amplitude, at most weakly nonlinear waves. Under typical ocean conditions, one finds $u_0/c_0 \sim 0.1$, indeed very small.

The expansion (12) may now be substituted into (7) and (4), and the Gaussian averages over $\mathbf{v}$ performed via Wick's theorem. For brevity we discuss only qualitative features. A full discussion, with applications to realistic ocean wave models is presented in [6]. The vertical profile of $\mathbf{u}$ and $\mathbf{D}$ may be quite complicated—depending on multiple products of $\hat{\mathbf{e}}(\mathbf{k},z)$ [7] in (2)—but overall amplitudes may be estimated. Since $\langle\mathbf{v}\rangle \equiv 0$, one finds $\mathbf{u} = O(u_0^2/c_0) \ll u_0$. Naively one expects $\mathbf{D} = O(u_0^2\tau)$;

however, the property $f(\mathbf{0}) = 0$ suppresses this term and one finds $\mathbf{D} = O[u_0^2\tau(u_0/c_0)^2]$ instead. For parameters appropriate to internal waves [so-called baroclinic inertia gravity (BIG) waves], one finds $\mathbf{u} \sim 5$ cm/s, and $\mathbf{D} \sim 10^4$ cm^2/s [6]. This value for D is insignificant on large (>100 km) scales, but can dominate eddy turbulent values on small (<10 km) scales, and hence may be important for so-called subgrid modeling of ocean dynamics. On the other hand, this value for $\mathbf{u}$ is *comparable to some ocean currents* and could be a significant source of coherent transport that has largely been overlooked.

We turn now to what may be the most physically interesting aspect of wave transport: fluctuations of the tracer concentration. We find that wave effects enter at *zeroth order* in u_0/c_0 and can therefore be very important. Spatial fluctuations of passive tracers, such as chlorophyll, serve as indicators of ocean dynamics in the top 10–20 m, traditionally thought to be dominated by 2D eddy turbulence. In their pioneering work, Gower *et al.* [8] suggested that the observed k^{-3} power-law spectrum of chlorophyll-*a* spatial fluctuations, Fig. 1, reflects the *kinetic energy* spectrum, which also follows a k^{-3} law in the direct enstrophy cascade region of the inertial range. This view was criticized in [9]: the tracer spectrum should actually follow that of the enstrophy, which exhibits a k^{-1} behavior. Observations in other areas of the ocean, Fig. 2, confirm this. The results in Fig. 1 therefore remained unexplained.

We now show that Fig. 1 can be explained by waves. In many regions, including high latitudes studied in [8], the relative level of eddy turbulence may be low, and the dynamics may be BIG wave dominated. Analyses of sea surface height (SSH) variations [10] confirm this. The insets of Figs. 1 and 2 illustrate the difference for two regions with, respectively, low and high levels of eddy turbulence. We now derive wave tracer spectra and show consistency with Fig. 1.

For simplicity, we consider an effective 2D *compressible* model ($\hat{d} = 2$, $\bar{d} = 0$) of the ocean near surface [11]. Let an initially waveless fluid have concentration field $\psi_0(\mathbf{r})$ and autocorrelation function $R_0(\mathbf{r} - \mathbf{r}') = [\psi_0(\mathbf{r})\psi_0(\mathbf{r}')]_{\rm av}$, in which $[\cdot]_{\rm av}$ denotes an ensemble average determined by transport processes *excluding* wave-induced motions. We use ψ_0 as an initial condition in (1), and compute

$$R(\mathbf{r} - \mathbf{r}') = \langle [\psi(\mathbf{r}, t)\psi(\mathbf{r}', t)]_{\rm av}\rangle, \tag{13}$$

which is time independent for $\tau \ll t \ll \tau_d$, where τ_d is the time scale on which diffusion and drift strongly alter ψ: we consider the short-term effects of the wave *pattern* on the concentration field which act before significant large-scale transport takes place. Using the formalism above one obtains

$$R(\mathbf{r}) = \int d^d r'\, \mathcal{K}(\mathbf{r}, \mathbf{r}') R_0(\mathbf{r}'),$$
$$\mathcal{K}(\mathbf{r}, \mathbf{r}') \equiv \int \frac{d^d k}{(2\pi)^d}\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}') - \sigma(\mathbf{k};\mathbf{r}')}, \tag{14}$$

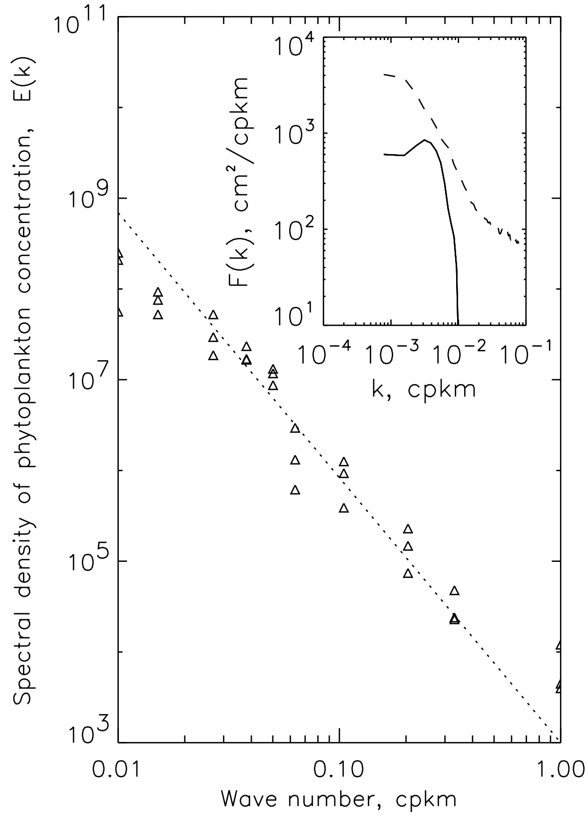

FIG. 1. Power spectrum of chlorophyll-*a* fluctuations in an 180 km × 250 km ocean area south of Iceland, as reported in [8] based on analysis of the Landsat multispectral imagery (reproduced courtesy of the authors). Triangles: experimental data. Dashed line: least squares $k^{-2.92}$ power-law fit. Inset: Power spectra of sea surface height fluctuations in this same region, based on the analysis of Topex/Poseidon ocean altimeter measurements described in [10]. Solid curve: Slow component of the SSH fluctuations associated with the vortical motions. Dashed curve: Fast component caused by the gravity-wave motions.

in which $\sigma(\mathbf{k}; \mathbf{r} - \mathbf{r}') = -\ln\langle e^{-i\mathbf{k}\cdot[\Delta\mathbf{Z}_{\mathbf{r}0}(t) - \Delta\mathbf{Z}_{\mathbf{r}'0}(t)]}\rangle$. The latter may be computed perturbatively using (12). However, for short-term effects, only the zeroth order result $\Delta\mathbf{Z}_{\mathbf{r}0}(t) = \int_0^t ds\, \mathbf{v}(\mathbf{x}, s)$ is required. One obtains then $\sigma(\mathbf{k}, \mathbf{r}) = \frac{1}{2}\sum_{i,j}[\Gamma_{ij}(\mathbf{r}) + \Gamma_{ji}(\mathbf{r})]k_i k_j$, with

$$\Gamma_{ij}(\mathbf{r}) = -\int_{-\infty}^{\infty} ds |s| [G_{ij}(\mathbf{0}, s) - G_{ij}(\mathbf{r}, s)], \tag{15}$$

in which $G_{ij}(\mathbf{r}, t) = \langle v_i(\mathbf{r}, t) v_j(\mathbf{0}, 0)\rangle$ ($\to 0$ for $t > \tau$) is the Eulerian velocity correlator. In deriving (15), a term $\propto t$ for $t > \tau$ actually vanishes due to $f(\mathbf{0}) = 0$. Corrections to (15) are of relative $O(u_0^2/c_0^2)$. In computing the Fourier transform $\hat{R}(\mathbf{k})$, this requires $k^2 u_0^4 \tau^2/c_0^2 \sim (k\lambda_0)^2 (u_0/c_0)^4 \ll 1$, i.e., $k \ll c_0^2/u_0^2 \lambda_0$. Since the inertial range of $\mathbf{v}$ corresponds roughly to $k > 1/\lambda_0$, the present calculation will be valid for k quite far inside it.

Within the above range one must consider two subranges: $1 < k\lambda_0 < c_0/u_0$ and $c_0/u_0 < k\lambda_0 < (c_0/u_0)^2$. In evaluating (14) these correspond to the small r

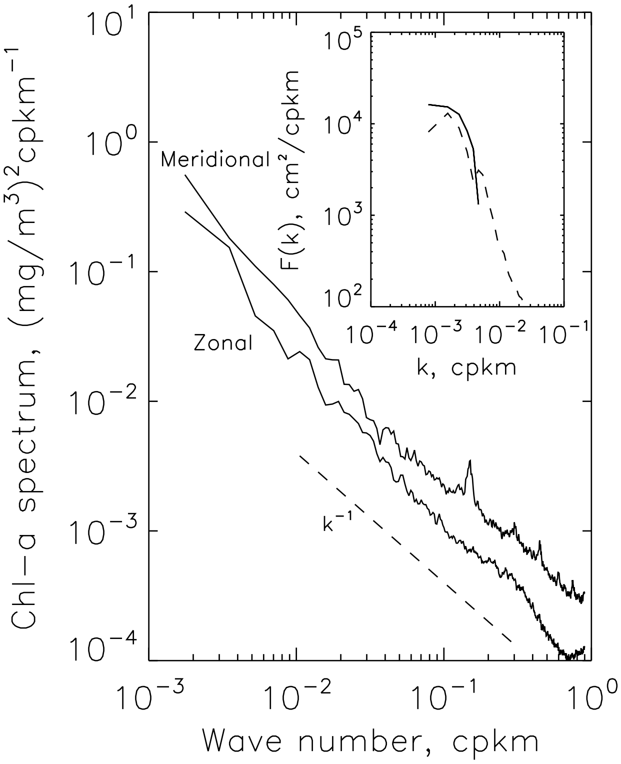

FIG. 2. Power spectrum of chlorophyll-a fluctuations in an ocean area east of Honshu Island (Japan), based on analysis of the OCTS multispectral imagery obtained from the ADEOS satellite (courtesy of the Japanese Space Agency NASDA). Dotted line: k^{-1} power law. Inset: Power spectra of sea surface height variations in this region, as described in the caption to Fig. 1.

asymptotics of $R(\mathbf{r})$ in the ranges $u_0/c_0 < r/\lambda_0 < 1$ and $(u_0/c_0)^2 < r/\lambda_0 < u_0/c_0$, respectively. In the first range, one is permitted to expand the exponential in (14), yielding, in Fourier space, $\hat{R}(k) = \hat{R}_0(k) + \Delta\hat{R}(k)$, with

$$\Delta R(\mathbf{k}) = \bar{\psi}^2 k^2 [F_L(\mathbf{k})/\omega(\mathbf{k})^2 + F_L(-\mathbf{k})/\omega(-\mathbf{k})^2], \quad (16)$$

where $\bar{\psi}$ is the overall mean concentration, and $F_L(k) \equiv \sum_{ij} k_i k_j F_{ij}(k)$ is the longitudinal (i.e., compressional part of the) wave-number spectrum $F_{ij}(\mathbf{k}) = f(\mathbf{k})\hat{e}_i(\mathbf{k})\hat{e}_j^*(\mathbf{k})$ [the Fourier transform of $G_{ij}(\mathbf{r}, 0)$] [12]. The waves therefore give an additive contribution to the background spectrum, observable then only if $\Delta\hat{R}(\mathbf{k}) \gtrsim \hat{R}_0(\mathbf{k})$, i.e., in regions of low eddy turbulence. For observed (angle integrated) spectrum $kF_L(\mathbf{k}) \sim k^{-3}$, and $\omega \sim c_0 k$ (appropriate to BIG waves shorter than the Rossby radius) one finds $k\Delta R(\mathbf{k}) \sim k^{-3}$. This subrange corresponds nicely to the 10^{-2}–10^{-1} km^{-1} range, and Eq. (16) is therefore completely consistent with Fig. 1.

In the second subrange, treating for simplicity only the isotropic case $\Gamma_{ij}(\mathbf{r}) = \Gamma(r)\delta_{ij}$, Eq. (14) may be reduced to the form (valid now for general d)

$$\mathcal{K}(r', |\mathbf{r}' - \mathbf{r}|) = [2\pi\Gamma(r')]^{-d/2} e^{-|\mathbf{r}'-\mathbf{r}|^2/2\Gamma(r')}. \quad (17)$$

The small r asymptotics of $R(r)$ may be analyzed and one finds the following: if $R_0(r) \approx R_0(0)[1 - Ar^\alpha]$, $\Gamma(r) \approx Br^\beta$, then $R(r) \approx R(0)[1 - Cr^\mu]$, with $\mu(\alpha, \beta) = 2(\alpha + d)/\beta - d$, corresponding to an angular integrated spectrum $k^{d-1}\hat{R}(k) \sim k^{-p}$ with

$$\begin{aligned} p(\alpha, \beta) &= \mu + d - (d - 1) \\ &= 2(\alpha + d)/\beta + 1 - d. \end{aligned} \quad (18)$$

Defining ζ, q via $\omega(k) \sim k^\zeta$ and $F_L(k) \sim k^{-d-q}$, Eq. (15) yields $\beta = \min\{q + 2\zeta, 2\}$. For $q > 0$, $\zeta = 1$, which includes BIG waves, one finds $\beta = 2$, and hence $\mu = \alpha$: the spectrum is unrenormalized in this regime. This remains to be tested by observational data.

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[6] P. B. Weichman and R. E. Glazman (to be published). **D** emerges as a certain weighted integral of the square of the velocity spectrum. For slowly decaying turbulent wave spectra these integrals could diverge, leading to scale-dependent transport familiar from Navier-Stokes turbulence. However, we find that wave spectra derived from both observation and weak turbulence theory are sufficiently steep to yield convergent integrals.
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[12] The result (16) may be derived more transparently by *linearizing* (1) via $\psi = \bar{\psi} + \delta\psi$, yielding $\partial_t \delta\psi = -\bar{\psi}\nabla \cdot \mathbf{v}$, whose solution yields precisely (16). It is clearly only the compressive part of $\mathbf{v}$ that changes the concentration ψ at linear order. This linearization is valid only in this first subrange of k.