Noise-Enhanced Phase Synchronization in Excitable Media

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We study the response of one- and two-dimensional excitable media to external spatiotemporal noise in terms of synchronization. The media are modeled by a finite-size lattice of locally coupled nonidentical units of the FitzHugh-Nagumo type driven by additive noise. We show that at nonzero noise level the behavior of the system becomes extremely ordered which is manifested by entrainment of the mean frequencies and by stochastic phase locking of distant oscillators in the lattice.

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Originally, synchronization has been studied for periodically driven self-sustained oscillators [1] and later noise sources were included into the consideration [2]. Today synchronization of noisy oscillators again achieves high importance in biomedical applications [3–5] and in neuroscience [6,7] as a possible mechanism of encoding of external signals [8]. Recent studies have shown that the class of systems and driving signals which exhibit synchronization could be significantly extended. Different types of synchronization have been found in chaotic systems [9–12]. Synchronization effects have been observed in stochastic systems, where noise controls the characteristic frequency of the system [13–16] and may enhance synchronization [17,18].

This positive, e.g., ordering, role of noise has been observed also in extended excitable systems, including spatiotemporal stochastic resonance [19], noise-induced spiral waves [20], noise-enhanced wave propagation [21,22], as well as noise-induced spiral chaos [23] and memory [24]. These effects have been observed in experiments with biological [25] and chemical [26] systems. In [27,28] the noise-induced synchronization of globally coupled identical neuron oscillators has been studied. Recently in Ref. [29] a new effect of noise-induced transition from pulsating spots to global oscillations in excitable media modeled by cellular automata has been reported. On the other hand, the authors of [30] studied synchronization of oscillators with randomly distributed natural frequencies.

Here we study phase synchronization in noisy excitable media which is modeled by a discrete network of diffusively coupled nonidentical FitzHugh-Nagumo oscillators. This model is generic for a number of applications in chemical physics and in neuroscience. In particular, it has been shown recently that the FitzHugh-Nagumo system models the firing activity of electroreceptor cells of the electric fish [31]. Thus, in contrast to previous studies our model combines local coupling, spatiotemporal noise, and nonidentity of the units. We will show that the space-time behavior of the system becomes extremely ordered at a nonzero noise level. With τ_u and τ_w being the characteristic time scales of the activator and inhibitor and introducing the dimensionless time by scaling the time as $\tau_w t$ it reads

$$\epsilon\dot{u}(t,n) = u - \frac{u^3}{3} - w + \gamma \sum_{n'} \frac{1}{2dl^2}[u(t,n') - u(t,n)], \quad (1)$$
$$\dot{w}(t,n) = u + a(n) + \sqrt{\frac{2D}{\tau_w l^d}}\,\xi(t,n),$$

where $\epsilon = \tau_u/\tau_w \ll 1$ and, hence, $u(t,n)$ and $w(t,n)$ are a fast and a slow variable, respectively. For the one-dimensional case, $d = 1$, these variables are defined on a chain $n = 1,\ldots,N$ while, in the two-dimensional case, $d = 2$, u and w are defined on a square lattice with the spacing l. The sum (over neighbors) stands for the discrete Laplace operator in one and two dimensions, respectively, modeling the local interactions with coupling strength $\gamma = D_u\tau_u$ and D_u is the diffusion coefficient of the activator. Further on we assume stochastic forcing by Gaussian white noise ξ, statistically independent in space and with zero mean $\langle\xi(t,n)\xi(t+\tau,m)\rangle = \delta_{m,n}\delta(\tau)$ [32].

Let us briefly review the local dynamics of the uncoupled element which depends significantly on parameters $a(n)$ responsible for excitory properties of the local dynamics. In the absence of noise for $a < 1$ the system possesses a stable periodic solution (limit cycle) generating a periodic sequence of spikes. The parameter value $a = 1$ corresponds to the birth of a limit cycle, and for $a > 1$ the single uncoupled element is excitable. The presence of stochastic perturbations leads to the existence of a peak in the spectral density even in the excitable regime. The peculiarity of excitable systems is that the coherence of noise-induced activity can be optimized by tuning the noise strength [33].

For a network of coupled FitzHugh-Nagumo oscillators it is natural to expect that for strong enough coupling the firing events of particular elements will be synchronized. In our numerical simulations we fixed $\epsilon = 0.01$, $\tau_w = 1$,

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$l = 1$, and $\gamma = 0.05$, while the activation parameters $a(n)$ are random numbers distributed uniformly on [1.03, 1.1]. This leads to a distribution of spiking times if noise is applied. We also use free boundary and random initial conditions. In the absence of noise any initial state of the system evolves to an equilibrium state.

Depending on noise strength D, for a sufficiently large value of the coupling strength three basic types of space-time behavior can be observed. For a small noise centers of excitation are nucleated very seldomly in random positions in the media giving rise to propagating waves. In this case different cells in the media are correlated only on a short time scale of a mean time of wave propagation and there is no synchronization between distant cells. For a large noise strength, the nucleation rate is very high and the media is represented by stochastically firing cells. However, for an optimal noise intensity the media becomes phase coherent: firings of different and distant cells occur almost in phase. Those three cases are shown in Fig. 1.

To characterize this effect for the one-dimensional case quantitatively we introduce the instantaneous phase $\Phi(t,n)$ of the nth element using the analytic signal representation [34]. We define the analytic signal $z(t,n) = u(t,n) + iy(t,n)$, where $y(t,n)$ is the Hilbert transform of the original variable $u(n,t)$ in the time domain: $y(t,n) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(\tau,n)}{t-\tau}d\tau$ and $\Phi(t,n) = \arctan[\frac{y(t,n)}{u(t,n)}]$. We choose the central cell in the media ($n = N/2$) as a reference element and then calculate the phase differences $\phi(t,k) = \Phi(t,N/2) - \Phi(t,N/2+k)$, $k = -N/2,\dots,N/2$. The results of calculations of the phase differences are shown in the inset of Fig. 2 for three values of noise intensity. For the optimal noise level the phases of different oscillators are locked during the time of computations. In the case of large distances between oscillators the phase fluctuations do indeed grow. Nevertheless, the phase difference is still bounded during long periods of time in a certain range. For nonoptimal noise intensities a partial phase synchronization with randomly occurring phase slips can be observed only between neighboring elements. For larger distances the diffusion of the phase differences becomes very strong and synchronization breaks down.

A measure of stochastic synchronization is the cross-diffusion coefficient [2] $D_{\mathrm{eff}}(k) = \frac{1}{2}\frac{d}{dt}[\langle\phi^2(t,k)\rangle - \langle\phi(t,k)\rangle^2]$. This quantity describes the spreading in time of an initial distribution of the phase difference between the $N/2$th and all other elements. If this diffusion constant decreases a longer phase locking epochs appears and, therefore, phase synchronization becomes stronger. A single measure is obtained by averaging $D_{\mathrm{eff}}(k)$ over the spatial distance $D_{\mathrm{eff}} = \frac{1}{N}\sum_{k=-N/2}^{N/2} D_{\mathrm{eff}}(k)$. The dependence of this averaged effective cross-diffusion constant versus noise intensity is in Fig. 2 and demonstrates a global minimum at nonzero noise level.

Synchronization is also defined as a frequency locking effect. In the case of a stochastic system one must use

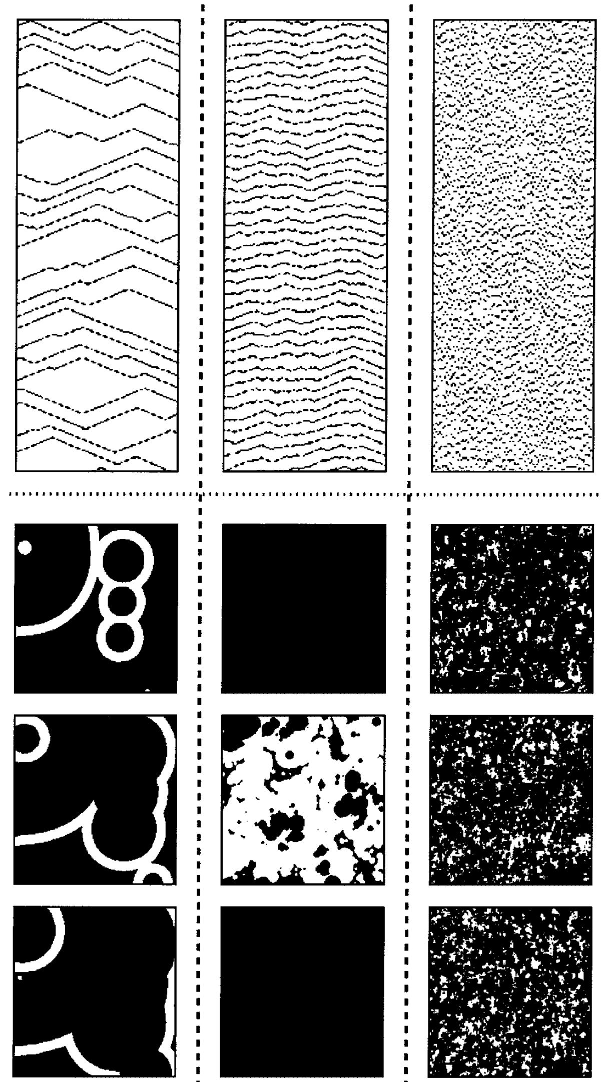

FIG. 1. Spatiotemporal evolution of the system described by Eq. (1) for different values of noise intensity (from left to right) $D = 1.125 \times 10^{-4}$; $D = 3.125 \times 10^{-4}$; $D = 5 \times 10^{-3}$. Other parameters: $\epsilon = 0.01$, $\gamma = 0.05$, and $l = 1$. The upper figure corresponds to the one-dimensional case with $N = 500$ elements; the vertical axes correspond to time and the horizontal line is space variable. Black dots indicate firing elements. The lower figure represents three frame sequences for the two-dimensional case of 200×200 lattice, where white dots correspond to firings.

the mean frequencies $\langle\omega(n)\rangle = \langle\dot{\Phi}(t,n)\rangle$ of the oscillators [2]. Because of the given distribution of the $a(n)$, the elements in the network have different randomly scattered frequencies for vanishing coupling. We have numerically built the distribution of the mean frequencies, calculated for every element across the network, $P(\langle\omega\rangle)$, for different noise intensities. The results are shown in Fig. 3. A remarkable effect of noise-enhanced space-time synchronization can be seen from this figure. For the optimal noise intensity, when the phases of different oscillators are locked for long periods of time, the mean frequencies are entrained and the distribution of

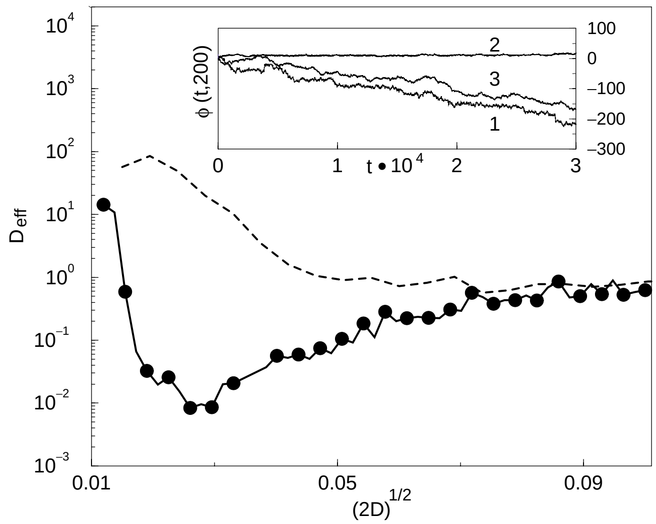

FIG. 2. The averaged effective cross-diffusion constant versus noise intensity. The dashed line corresponds to the uncoupled lattice ($\gamma = 0$). Inset: Phase differences between indicated oscillators for different values of noise variance: $D = 1.125 \times 10^{-4}$ (1); $D = 3.125 \times 10^{-4}$ (2); $D = 5 \times 10^{-3}$ (3). Other parameters are the same as in the previous figure.

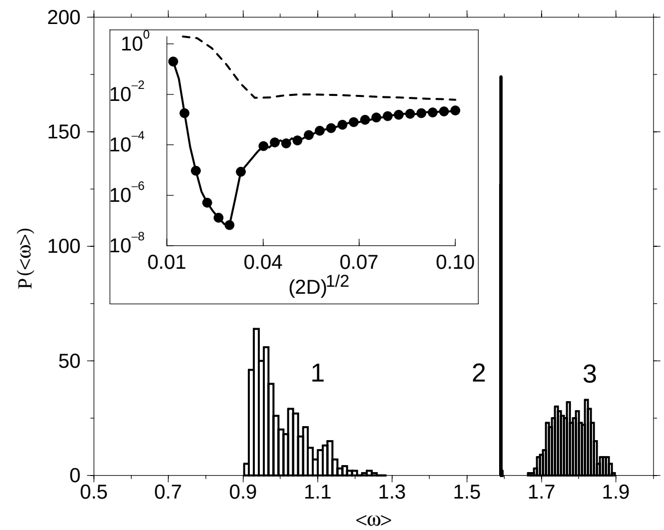

FIG. 3. Distribution of the mean frequencies of oscillators for different values of noise intensity: $D = 1.125 \times 10^{-4}$ (1); $D = 3.125 \times 10^{-4}$ (2); $D = 5 \times 10^{-3}$ (3). Inset: the variance of the mean frequencies $\sigma^2_{\langle\omega\rangle}$ versus noise intensity. Other parameters are the same as in Fig. 1. The dashed line corresponds to uncoupled lattice ($\gamma = 0$).

the mean frequencies becomes extremely narrow. For nonoptimal noises the mean frequencies show rather wide distributions indicating the lack of synchronization. The last figure clearly indicates noise-induced space-time ordering in the system based on a synchronization mechanism. We quantify this effect further by calculating the mean square deviation $\sigma_{\langle\omega\rangle}$ of the mean frequencies averaging over the network $\sigma^2_{\langle\omega\rangle} = \frac{1}{N}\sum_{n=1}^{N}\langle\omega(n)^2\rangle - [\frac{1}{N}\sum_{n=1}^{N}\langle\omega(n)\rangle]^2$. The dependence of this quantity on noise intensity is shown in the inset of Fig. 3. It is seen that $\sigma^2_{\langle\omega\rangle}$ can be minimized up to very small values (of the order of 10^{-7}) by tuning the noise. Synchronization is indeed absent for vanishing coupling. The mean firing rate of the cells saturates for large D, leading to the saturation of $\sigma_{\langle\omega\rangle}$ and also the effective cross-diffusion coefficient (shown by the dashed lines in Figs. 3 and 2, respectively).

With the increase of the system size phase synchronization becomes worse as the phase fluctuation grows with the growth of the system size. This is in agreement with the theoretical predictions of Ref. [35]. Two-dimensional excitable media (see lower part of Fig. 1) demonstrate qualitatively the same behavior (in the sense of the averaged characteristic described above). For a weak noise the system possesses noise-induced target waves which are initiated in random positions in the media. Collapsing of such waves cannot exhibit stable spiral waves since the velocity of the waves at the intersection is always directed outside of the intersecting region. Therefore no new open spirals may occur. However, in the case of parametric noise the propagating fronts may locally backfire small directed spots which break propagating excitations and make spirals possible [23]. At the optimal noise level the whole media oscillates nearly periodically (see the middle row in the lower part of Fig. 1). Finally, the case of large noise is represented by randomly flushing clusters.

The mechanism of the noise-induced synchronization is rooted in the behavior of a single uncoupled element. The noise-induced oscillations are most coherent at a nonzero noise intensity and the quality factor of the noise-induced peak in the power spectrum is maximal. In this regime the mean firing rate (or the mean frequency) of the system approaches the peak frequency in the power spectrum. In the case of weak noise the mean firing rate depends exponentially on the control parameter a ($a > 1$) [36]. However, with the increase of D the mean frequency approaches the value of the peak frequency in the power spectrum and then saturates. That is why with the increase of noise from a very low level the mismatch between characteristic frequencies of elements in the coupled array decreases providing better conditions for mutual synchronization. On the other hand, the noise-induced oscillation becomes more coherent. These effects will tend to facilitate synchronization among elements in the network. Large noise, alternatively, destroys again the coherence of local stochastic oscillations (the frequency and phase fluctuations grow rapidly) and also leads to the destruction of spatial coherent structures. The optimal noise intensity at which synchronization is most pronounced depends on the range of the distribution of activation parameters $a(n)$: with the increase of the range of disorder the optimal noise intensity shifts towards smaller values.

In conclusion, we have demonstrated the space-time stochastic synchronization in discrete excitable media modeled by diffusively coupled FitzHugh-Nagumo oscillators driven by external noise. This phenomenon manifests itself as stochastic phase locking between distant cells and also as the mean frequency entrainment. We have also revealed that the degree of phase coherence, quantified by the effective diffusion constant and by the variance of the mean frequencies of oscillators, can be enhanced by tuning the noise intensity. Noise contaminated oscillations and synchronization are rather typical in biological systems, starting from molecular [37] and cell networks [5] up to the dynamics of biological populations [38]. Our results indicate that noise might play an important role in the self-organization of coupled excitable elements through the synchronization mechanism. Therefore, our findings could be of importance for biological/medical applications.

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