One-Loop UV Divergent Structure of U(1) Yang-Mills Theory on Noncommutative $\mathbb{R}^4$

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We show that U(1) Yang-Mills theory on noncommutative $\mathbb{R}^4$ can be renormalized at the one-loop level by multiplicative dimensional renormalization of the coupling constant and fields of the theory. We compute the beta function of the theory and conclude that the theory is asymptotically free. We also show that the Weyl-Moyal matrix defining the deformed product over the space of functions on $\mathbb{R}^4$ is not renormalized at the one-loop level.

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Field theories on noncommutative spaces may play an important role in unraveling the properties of nature at the Planck scale [1,2]. Yang-Mills theories on the noncommutative torus occur in compactifications of M theory [2], and there is already a good many papers where M theory on noncommutative tori has been studied [3]. Still, it has not been established yet that these field theories are well defined at the quantum level. Although a few initial steps were taken in Ref. [1] (see also [4]), the very definition of quantum field theory over noncommutative spaces (the analog of the standard Wightman-Osterwalder-Schrader axioms, scattering theory, etc.) is yet to be stated. Quantum field theories on fuzzy spheres [5] had been studied previously and its UV finiteness established [6]. Here the number of quantum degrees of freedom is finite and hence it seems that the quantum theory exists. This is not clear at all for field theories over noncommutative spaces such as the noncommutative $\mathbb{R}^4$ [7], the noncommutative 3-tori [8], or the noncommutative plane [9]. In these cases the quantum field theory has UV divergences and there remains the question whether these theories are renormalizable. It should be noticed that for the theories at hand the interaction terms in Fourier space are no longer polynomials in the momenta, and hence it cannot be taken for granted that the renormalization program (either perturbative or nonperturbative) works.

The purpose of this Letter is to analyze the one-loop UV divergent structure of the simplest pure gauge theory over the noncommutative $\mathbb{R}^4$. This theory is a U(1) theory but it is not a free theory due to the noncommutative character of the base space. Indeed, now the field strength (curvature) is no longer linear in the gauge field (connection).

Field equations over the noncommutative $\mathbb{R}^4$ can be represented as equations over a deformation of the C^* algebra $C_\infty(\mathbb{R}^4)$ of continuous complex-valued functions over $\mathbb{R}^4$ vanishing at infinity [10]. This deformation is given by the Weyl product

$$(f \star g)(x) = \iint \frac{d^4q}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\, e^{i(q+p)x} e^{i\omega_{\mu\nu}q_\mu p_\nu} f(q) g(p)\,.$$

Here $f(q)$ and $q(p)$ are, respectively, the Fourier transforms of $f(x)$ and $g(x)$, and $\omega_{\mu\nu}$ is a constant antisymmetric matrix of rank four. Connes' noncommutative geometry formalism gives mathematical rigor to the concept of classical U(1) gauge field over such a noncommutative space [11,12]. This gauge field is provided by a real vector function, $A_\mu(x)$, on $\mathbb{R}^4$. The field strength $F_{\mu\nu}$ for this gauge field now reads $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\{A_\mu, A_\nu\}$, where $\{A_\mu, A_\nu\}(x) = (A_\mu \star A_\nu)(x) - (A_\nu \star A_\mu)(x)$ is the Moyal bracket.

The classical U(1) field theory over noncommutative $\mathbb{R}^4$ is given by the Yang-Mills functional

$$S_{\rm YM} = \frac{1}{4g^2}\int (F_{\mu\nu} \star F_{\mu\nu})(x)\,, \tag{1}$$

where $F_{\mu\nu}(x)$ is given above. This action is invariant under gauge transformations which have the following infinitesimal form: $\delta A_\mu(x) = \partial_\mu\theta + i\{A_\mu, \theta\}(x)$.

The next task to tackle is the construction of a quantum field theory whose classical counterpart is the previous theory. What we mean by a quantum field theory over a noncommutative space is by no means obvious, e.g., which mathematical objects define the quantum physics have not been properly established yet (see Ref. [9] for discussions on this point). In this paper we shall assume that the quantum theory is defined by the Green functions of the theory, i.e., by the generating functional

$$Z[j] = N\int \mathcal{D}\phi(x) e^{-S+\int d^4x j(x)\phi(x)} \equiv N e^{-S_{\rm int}[\delta/\delta j(x)]}\exp\left[\frac{1}{2}\iint d^4x d^4y\, j(x)P(x-y)j(y)\right],$$

where ϕ denotes generically the "fields" of the theory and $P(x-y)$ denotes the inverse of the kinetic term (quadratic in the fields) in S. S and $S_{\rm int}$ denote, respectively, the classical action over the noncommutative space in question

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and the interaction terms in S. The previous definition of $Z[j]$ is to be understood as a formal expansion in terms of Feynman diagrams.

Since our Yang-Mills action is invariant under gauge transformations, its kinetic term has no inverse and a gauge-fixing term has to be introduced. We shall do this in a consistent way by using the Becchi-Rouet-Stora (BRS) formalism. Let us introduce the ghost fields c and $\bar{c}$, the gauge-fixing field B, and define the BRS transformations as follows:

$$sA_\mu(x) = D_\mu c(x) = \partial_\mu c(x) + i\{A_\mu, c\}(x),\ s\bar{c}(x) = B,$$

$$sB(x) = 0,\ sc(x) = -(c \star c)(x).$$

To keep track of the renormalization of the composite transformations $sA_\mu(x)$ and $sc(x)$, one also introduces the external fields $J_\mu(x)$ and $H(x)$ which couple to them [13]. The BRS invariant classical four-dimensional action is

$$S_{cl} = S_{\rm YM} + S_{gf} + S_{\rm ext}, \qquad (2)$$

where $S_{\rm YM}$ has been given in Eq. (1) and

$$S_{\rm gf} = \int d^4x \hat{s}\left[\bar{c} \star \left(\frac{\alpha}{2} B - \partial_\mu A^\mu\right)\right](x),$$

$$S_{\rm ext} = \int d^4x \left(J^\mu \star sA_\mu + H \star sc\right)(x).$$

Now, standard path integral formal manipulations lead to the Slavnov-Taylor identity for the 1PI functional $\Gamma[A_\mu, B, c, \bar{c}; J_\mu, H]$. This identity reads

$$S(\Gamma) \equiv \int d^4x \left[\frac{\delta\Gamma}{\delta J^\mu}\frac{\delta\Gamma}{\delta A_\mu} + \frac{\delta\Gamma}{\delta H}\frac{\delta\Gamma}{\delta c} + B\frac{\delta\Gamma}{\delta\bar{c}}\right] = 0. \qquad (3)$$

We shall see in this paper that, as far as our explicit computations reach, no anomalies occur in the quantum theory and that, indeed, a renormalized BRS invariant 1PI functional does exist. We shall be working only at the one-loop level.

It is not difficult to see that one-loop UV divergent 1PI Green functions are the following: Γ_{AA}, Γ_{AAA}, Γ_{AAAA}, $\Gamma_{\bar{c}c}$, $\Gamma_{\bar{c}Ac}$, Γ_{Jc}, Γ_{JAc}, and Γ_{Hcc}, with obvious notation. Notice that the Feynman rules from Eq. (2) are obtained from the Feynman rules for Su(N) Yang-Mills theory on commutative Euclidean space upon the replacement $f_{a_1a_2a_3} \to 2\sin\omega(p_2,p_3)$, p_2 and p_3 being, respectively, the momenta carried by the lines with color index a_2 and a_3. Hence, the counterpart in our U(1) theory of a Feynman diagram, which is finite by power counting in the standard SU(N) theory, will also be finite. Since no action principle (see Ref. [13], and references therein) has been shown to hold yet for field theories over noncommutative spaces, we cannot carry out a cohomological study of the renormalizability of the theory at hand. We shall proceed by performing explicit computations.

To regularize the Feynman integrals of our theory will shall use dimensional regularization. The dimensionally regularized counterpart of any Feynman diagram is defined as follows: First, $2i\sin\omega(p_2,p_3)$ is expressed as $e^{i\omega(p_2,p_3)} - e^{-i\omega(p_2,p_3)}$; second, the four-dimensional momentum measure $d^4p/(2\pi)^4$ is replaced with the "D-dimensional" measure $d^Dp/(2\pi)^D$ and any four-dimensional algebraic expression with a D-dimensional one defined according to the rules in Ref. [14]; third, Gaussian integration over the D-dimensional loop momenta is carried out, which leads to an integral over the α-parameter space of Schwinger (see [15]); fourth, D is promoted to a complex variable, and any formal expression is defined to be an algebraic expression satisfying only the algebraic rules in Ref. [14], with the Weyl-Moyal matrix being "intrinsically four dimensional."

We have computed the one-loop UV divergent contribution to all the divergent 1PI Green functions. The results we have obtained thus read

$$\Gamma^{(AA)}_{\mu_1\mu_2}(p) = -\left(\frac{1}{(4\pi)^2\varepsilon}\right)\left(\frac{13}{3} - \alpha\right)\left(p_{\mu_1}p_{\mu_2} - p^2 g_{\mu_1\mu_2}\right), \qquad (4a)$$

$$\Gamma^{(AAA)}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) = -i\left(\frac{1}{(4\pi)^2\varepsilon}\right)\left(\frac{17}{3} - 3\alpha\right)\sin[\omega_{\mu\nu}(p_2)_\mu(p_3)_\nu]$$
$$\times [(p_1 - p_3)_{\mu_2}g_{\mu_1\mu_3} + (p_2 - p_1)_{\mu_3}g_{\mu_1\mu_2} + (p_3 - p_2)_{\mu_1}g_{\mu_2\mu_3}], \qquad (4b)$$

$$\Gamma^{(AAAA)}_{\mu_1\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4) = \left(\frac{1}{(4\pi)^2\varepsilon}\right)\left(\frac{4}{3} - 2\alpha\right)$$
$$\times 4\{\sin[\omega_{\mu\nu}(p_1)_\mu(p_2)_\nu]\sin[\omega_{\mu\nu}(p_3)_\mu(p_4)_\nu](g_{\mu_1\mu_3}g_{\mu_2\mu_4} - g_{\mu_1\mu_4}g_{\mu_2\mu_3})$$
$$+ \sin[\omega_{\mu\nu}(p_1)_\mu(p_4)_\nu]\sin[\omega_{\mu\nu}(p_3)_\mu(p_2)_\nu](g_{\mu_2\mu_4}g_{\mu_1\mu_3} - g_{\mu_4\mu_3}g_{\mu_1\mu_2})$$
$$+ \sin[\omega_{\mu\nu}(p_4)_\mu(p_2)_\nu]\sin[\omega_{\mu\nu}(p_3)_\mu(p_1)_\nu](g_{\mu_4\mu_3}g_{\mu_2\mu_1} - g_{\mu_1\mu_4}g_{\mu_2\mu_3})\}, \qquad (4c)$$

$$\Gamma^{(\bar{c}c)}(p) = -\left(\frac{1}{(4\pi)^2\varepsilon}\right)\left(\frac{1}{2}\right)(3-\alpha)g^2p^2, \tag{4d}$$

$$\Gamma^{(\bar{c}Ac)}_{\mu_2}(p_1,p_2,p_3) = -i\left(\frac{1}{(4\pi)^2\varepsilon}\right)(\alpha g^2)(p_1)_{\mu_2}2\sin[\omega_{\mu\nu}(p_2)_\mu(p_3)_\nu], \tag{4e}$$

$$\Gamma^{(Jc)}_{\mu}(p) = \left(\frac{1}{(4\pi)^2\varepsilon}\right)\left(\frac{3-\alpha}{2}\right)g^2p_\mu, \tag{4f}$$

$$\Gamma^{(JAc)}_{\mu_1\mu_2}(p_1,p_2,p_3) = \left(\frac{1}{(4\pi)^2\varepsilon}\right)(\alpha g^2)g_{\mu_1\mu_2}2\sin[\omega_{\mu\nu}(p_2)_\mu(p_3)_\nu], \tag{4g}$$

$$\Gamma^{(Hcc)}(p_1,p_2,p_3) = -\left(\frac{1}{(4\pi)^2\varepsilon}\right)(\alpha g^2)e^{i[\omega_{\mu\nu}(p_2)_\mu(p_3)_\nu]}, \tag{4h}$$

where $D = 4 - 2\varepsilon$. Notice that the momentum structure of the previous contributions is the same as the corresponding term in the BRS action in Eq. (2). So, one would expect that these 1PI contributions can be subtracted by MS multiplicative renormalization of the fields and parameters in the BRS invariant action. And, indeed, this is so, if we perform the following infinite renormalizations: $g_0 = \mu^{2\varepsilon}Z_g g$, $\alpha_0 = Z_\alpha\alpha$, $A_{0\mu} = Z_A A_\mu$, $B_0 = Z_B B$, $J_{0\mu} = Z_J J_\mu$, $H_0 = Z_H H$, $c_0 = Z_c c$, and $\bar{c}_0 = Z_{\bar{c}}\bar{c}$. Here the subscript 0 labels the bare quantities and

$$Z_g = 1 - \frac{1}{(4\pi)^2\varepsilon}\frac{22}{3}g^2,\; Z_A = 1 - \frac{1}{(4\pi)^2\varepsilon}\frac{3+\alpha}{2}g^2,$$
$$Z_{\bar{c}}Z_c = 1 + \frac{1}{(4\pi)^2\varepsilon}\frac{3-\alpha}{2}g^2,\; Z_{\bar{c}}Z_A Z_c = 1 - \frac{1}{(4\pi)^2\varepsilon}\alpha g^2,$$
$$Z_H Z_c^2 = 1 - \frac{1}{(4\pi)^2\varepsilon}\alpha g^2,\; Z_B = Z_A^{-1},\; Z_\alpha = Z_A^2, \quad \text{and} \quad Z_J = Z_{\bar{c}}. \tag{5}$$

Notice that there is no renormalization of the matrix $\omega_{\mu\nu}$. That these Z's render UV-finite the 1PI functions whose pole contribution is in Eqs. (4) is a consequence of BRS invariance. Indeed, in view of Eqs. (4), it is not difficult to show that the singular contribution $\Gamma^{(\text{pole})}$ to the dimensionally regularized 1PI functional can be recast into the form

$$\Gamma^{(\text{pole})} = \frac{a}{4g^2}\int d^Dx(F_{\mu\nu}\star F_{\mu\nu})(x) + \mathcal{B}_D X, \tag{6}$$

where

$$X = \int d^Dx[a_1(J_\mu - \partial_\mu\bar{c})\star A_\mu - a_2 H\star c](x),$$
$$a = \frac{1}{(4\pi)^2\varepsilon}\frac{22}{3}g^2,\; a_1 = -\frac{1}{(4\pi)^2\varepsilon}\frac{3+\alpha}{2}g^2,$$
$$a_2 = -\frac{1}{(4\pi)^2\varepsilon}\alpha g^2,$$

and $\mathcal{B}_D$ is the linearized Slavnov-Taylor operator acting upon the space of formal algebraic expressions constructed with D-dimensional monomials of the fields and their derivatives. $\mathcal{B}_D$ is defined as follows:

$$\mathcal{B}_D = \int d^Dx\left[\frac{\delta S_{cl}}{\delta J^\mu}\frac{\delta}{\delta A_\mu} + \frac{\delta S_{cl}}{\delta A^\mu}\frac{\delta}{\delta J_\mu} + \frac{\delta S_{cl}}{\delta H}\frac{\delta}{\delta c} + \frac{\delta S_{cl}}{\delta c}\frac{\delta}{\delta H} + B\frac{\delta}{\delta\bar{c}}\right].$$

The conclusion that one draws from Eq. (6) is that the UV divergent contributions displayed in Eqs. (4) are BRS invariant. Notice that $\Gamma^{(\text{pole})}$ is the sum of two terms: the second is $\mathcal{B}_D$ exact (recall that $\mathcal{B}_D^2 = 0$), whereas the first, the Yang-Mills term, is $\mathcal{B}_D$ closed. This all goes hand-in-hand with the analysis of the UV divergent contributions in standard SU(N) Yang-Mills theory. And, indeed, as in standard four-dimensional Yang-Mills theory, we have $Z_g = 1 - a$, $Z_A = 1 + a_1$, $Z_{\bar{c}}Z_c = 1 - a_1 + a_2$, $Z_{\bar{c}}Z_A Z_c = 1 + a_2$, $Z_H Z_c^2 = 1 + a_2$, $Z_B = Z_A^{-1}$, $Z_\alpha = Z_A^2$, and $Z_J = Z_{\bar{c}}$. Equation (5) is thus recovered. Let us remark that the values we have obtained for the Z's agree with the corresponding values of the Z's of standard SU(N) Yang-Mills theory on commutative $\mathbb{R}^4$ upon replacing in the latter the constant $C_2(G)$ (the quadratic Casimir in the adjoint representation) with 2. Actually, the UV divergent contributions in Eqs. (4) agree with those in the standard SU(N) Yang-Mills theory, if the following substitutions are made in the latter: $f_{a_1a_2a_3} \to 2\sin\omega(p_2,p_3)$ and $C_2(G) \to 2$. Recall that the structure constants are not renormalized in standard SU(N) Yang-Mills theory; $\omega_{\mu\nu}$, the matrix defining the Weyl-Moyal product, is not renormalized either.

We shall define the order $\hbar$ MS renormalized 1PI functional $\Gamma^{(1),\text{MS}}_{\text{ren}}$ as usual:

$$\Gamma^{(1),\text{MS}}_{\text{ren}} = \lim_{\varepsilon\to 0}[\Gamma^{(1)}_{D\text{reg}} - \Gamma^{(\text{pole})}],$$

where $\Gamma^{(1)}_{D\mathrm{reg}}$ denotes the dimensionally regularized 1PI functional at order $\hbar$, and $\Gamma^{(\mathrm{pole})}$ is given in Eqs. (4). The limit $\varepsilon \to 0$ is taken after performing the subtraction of the pole and replacing every D-dimensional algebraic object with its four-dimensional counterpart [14]; this is why we have denoted it by lim.

Since $\Gamma^{(1)}_{D\mathrm{reg}}$ is BRS invariant, i.e., $\mathcal{B}_D \Gamma^{(1)}_{D\mathrm{reg}} = 0$, the MS renormalized 1PI functional $\Gamma^{(1),\mathrm{MS}}_{\mathrm{ren}}$ is also BRS invariant: $\mathcal{B}\Gamma^{(1),\mathrm{MS}}_{\mathrm{ren}} = 0$. The operator $\mathcal{B}$ is the counterpart of $\mathcal{B}_D$ at $D = 4$: the linearized Slavnov-Taylor operator in noncommutative $\mathbb{R}^4$. We thus conclude that the Slavnov-Taylor identity [Eq. (3)] holds for the renormalized theory at order $\hbar$. This statement is not completely rigorous since there is no proof as yet that the quantum action principle [14] holds for the dimensionally regularized amplitudes of the theory at hand. However, in our computations we have found no hint that this principle might not be valid here.

By using standard textbook techniques, one can work out the renormalization group equation for $\Gamma^{\mathrm{MS}}_{\mathrm{ren}}$:

$$\left[\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} + \delta_\alpha \frac{\partial}{\partial \alpha} - \sum_\phi \gamma_\phi \int d^4x\, \phi(x) \frac{\delta}{\delta \phi(x)}\right] \Gamma^{\mathrm{MS}}_{\mathrm{ren}}[\phi; g, \omega, \alpha; \mu] = 0 .$$

The fields are denoted by ϕ. It should be noticed that $\omega_{\mu\nu}$ is a dimensionful parameter which does not run. The one-loop beta function of the theory is easily computed to be

$$\beta(g^2) \equiv \mu \frac{dg^2}{d\mu} = -\frac{1}{8\pi^2} \frac{22}{3} g^4 .$$

Hence, the theory is asymptotically free. The other renormalization group coefficients read, at the one-loop level,

$$\gamma_A = +\frac{1}{8\pi^2}\left(\frac{3 + \alpha}{2}\right) g^4, \qquad \gamma_c = +\frac{1}{8\pi^2}\, \alpha g^4,$$
$$\gamma_J = \gamma_{\bar{c}} = \gamma_B = -\gamma_A, \qquad \delta_\alpha = -2\gamma_A \alpha,$$
$$\gamma_H = -\gamma_c .$$

We shall conclude with two remarks. First, the structure of the UV divergences, which is not a polynomial in momentum space, is a polynomial in the fields and their derivatives with respect to the Weyl product. One wonders whether this generalizes to higher loops upon subtraction of subdivergences and whether the theory of normal products (on which the method of algebraic renormalization rests) remains valid upon replacing the ordinary product with the Weyl product. Second, $\Gamma^{(\mathrm{pole})}$ verifies both the gauge-fixing equation and the ghost equation (see Ref. [13] for definitions) and, hence, so does the MS renormalized 1PI functional up to order $\hbar$.

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