New Type of Intensity Correlation in Random Media

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A monochromatic point source, embedded in a three-dimensional disordered medium, is considered. The resulting intensity pattern exhibits a new type of long-range correlation. The range of these correlations is infinite and their magnitude, normalized to the average intensity, is of order $1/k_0\ell$, where k_0 and ℓ are the wave number and the mean free path, respectively.

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A wave propagating in a random medium undergoes multiple scattering and produces a complicated, irregular intensity pattern. Such a pattern is described in statistical terms and one of its important characteristics is the correlation function $\langle I(\vec{r}_1)I(\vec{r}_2)\rangle$, where $I(\vec{r})$ is the wave intensity at point $\vec{r}$ and the angular brackets designate the ensemble average. This correlation function has been studied in recent years, both theoretically and experimentally, and various types of correlations have been identified [1]. There are large short-range correlations, due to interference among the waves arriving, via all possible scattering sequences, to a neighborhood of a given point; this interference is responsible for the large spatial fluctuations in intensity (speckles). There exist also weak long-range correlations, due to diffusion which propagates the locally large fluctuations to distant regions in space. The purpose of this Letter is to identify and study a new type of long-range correlation which, under appropriate circumstances, can dominate the "standard" long-range correlations [1].

I consider a monochromatic point source embedded in an infinite three-dimensional random medium. The position of the source is $\vec{r}_0$ and its frequency is ω. Assuming a scalar wave, one has the following equation for the field (the Green's function) at point $\vec{r}$:

$$\{\nabla^2 + k_0^2[1 + \mu(\vec{r})] + i\eta\}G_\omega(\vec{r},\vec{r}_0) = \delta(\vec{r} - \vec{r}_0)\,. \tag{1}$$

Here $\mu(\vec{r})$ is the fractional fluctuation of the dielectric constant, η is a positive infinitesimal, and $k_0 = \omega/c$, where c is the speed of propagation in the average medium. I assume that $\mu(\vec{r})$ obeys white-noise Gaussian statistics, i.e.,

$$\langle \mu(\vec{r})\mu(\vec{r}\,')\rangle = u\delta(\vec{r} - \vec{r}\,')\,, \tag{2}$$

where the constant u describes the strength of the disorder.

For weak disorder, the average field, the average intensity, and various correlation functions can be computed by the diagram technique [1]. The average field, $\langle G_\omega(\vec{r},\vec{r}_0)\rangle$, decays exponentially away from the source:

$$\langle G_\omega(\vec{r},\vec{r}_0)\rangle = -\frac{1}{4\pi|\vec{r} - \vec{r}_0|}\exp\left[\left(ik_0 - \frac{1}{2\ell}\right)|\vec{r} - \vec{r}_0|\right], \tag{3}$$

where $\ell = 4\pi/uk_0^4$ is the mean free path and $k_0\ell \gg 1$.

The intensity at point $\vec{r}$ is defined as $I_\omega(\vec{r}) = |G_\omega(\vec{r},\vec{r}_0)|^2$ and its average value is given by the diagram in Fig. 1. Intensity propagates from the source to a distant observation point, such that $|\vec{r} - \vec{r}_0| \gg \ell$, by a diffusion process. This is represented by a diffusion ladder (the shaded box), $T(\vec{r}_1,\vec{r}_2) = 3/(\ell^3|\vec{r}_1 - \vec{r}_2|)$. The vertices, connecting the external points to the ladder, are short-range objects which can be replaced by the number

$$\int d^3r_1\,\langle G_\omega(\vec{r}_0,\vec{r}_1)\rangle\langle G^*_\omega(\vec{r}_0,\vec{r}_1)\rangle = \frac{\ell}{4\pi}\,, \tag{4}$$

so that the average intensity at point $\vec{r}$ is

$$\langle I_\omega(\vec{r})\rangle = \left(\frac{\ell}{4\pi}\right)^2 T(\vec{r},\vec{r}_0) = \frac{3}{16\pi^2\ell|\vec{r} - \vec{r}_0|}\,. \tag{5}$$

Let us now consider the correlation function

$$\frac{\langle \Delta I_\omega(\vec{r})\Delta I_\omega(\vec{r} + \Delta\vec{r})\rangle}{\langle I_\omega(\vec{r})\rangle\langle I_\omega(\vec{r} + \Delta\vec{r})\rangle} \equiv C(\Delta r)\,, \tag{6}$$

where $\Delta I_\omega(\vec{r}) = I_\omega(\vec{r}) - \langle I_\omega(\vec{r})\rangle$ is the deviation from the average value. The points $\vec{r}$ and $\vec{r} + \Delta\vec{r}$ are assumed to be far from the source, i.e., many mean free paths away. Since only long-range correlations are considered here, $\Delta r \gg \ell$ is assumed.

The leading contribution to the long-range correlations is represented by the diagram in Fig. 2. The diagram contains two pairs of Green's functions. One pair propagates, via a diffusion ladder, to point $\vec{r}$; the other, to point $\vec{r} + \Delta\vec{r}$. The pairs are connected by a dashed line, which makes this diagram belong to the set of diagrams contributing to $C(\Delta r)$. The physical interpretation of this diagram is straightforward: the wave, emanating from the source, gets scattered at some point $\vec{r}_1$, close to the source. The secondary wave, emerging from $\vec{r}_1$, propagates by

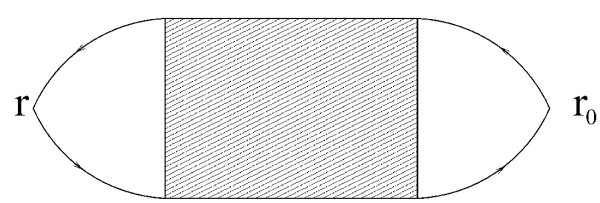

FIG. 1. The average intensity.

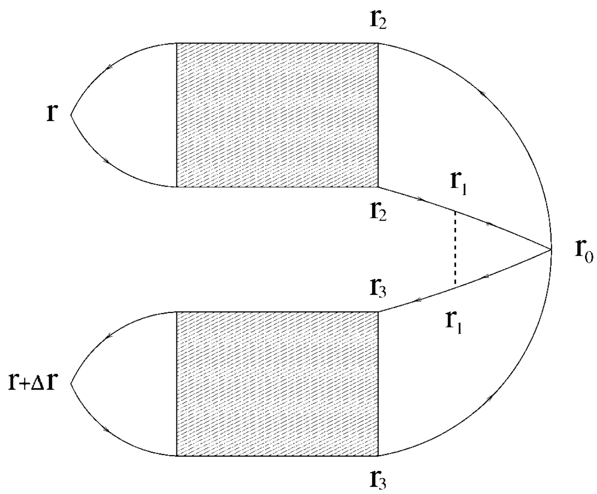

FIG. 2. The correlation function (C_0 term).

diffusion to two distant points, $\vec{r}$ and $\vec{r} + \Delta\vec{r}$. In this way intensity correlation at the two points is established.

In order to evaluate the diagram in Fig. 2 one has to compute the vertex, depicted separately in Fig. 3. The vertex is a short-range object and, thus, can be replaced by a point, located at $\vec{r}_0$, from which the two diffusion ladders emerge. The number V, assigned to the point is

$$V = \frac{4\pi}{\ell}\int d^3r_1\, d^3r_2\, d^3r_3 \langle G_\omega(\vec{r}_2 - \vec{r}_0)\rangle \times \langle G^*_\omega(\vec{r}_2 - \vec{r}_1)\rangle \langle G^*_\omega(\vec{r}_1 - \vec{r}_0)\rangle \langle G_\omega(\vec{r}_3 - \vec{r}_1)\rangle \times \langle G_\omega(\vec{r}_1 - \vec{r}_0)\rangle \langle G^*_\omega(\vec{r}_3 - \vec{r}_0)\rangle . \quad (7)$$

Integration over $\vec{r}_2$ gives

$$\int d^3r_2 \langle G_\omega(\vec{r}_2 - \vec{r}_0)\rangle \langle G^*_\omega(\vec{r}_2 - \vec{r}_1)\rangle = \frac{\ell}{4\pi} \times f_\omega(|\vec{r}_0 - \vec{r}_1|), \quad (8)$$

where

$$f_\omega(x) = \frac{\sin k_0 x}{k_0 x}\exp\left(-\frac{x}{2\ell}\right). \quad (9)$$

Integration over $\vec{r}_3$ gives an identical contribution, so that

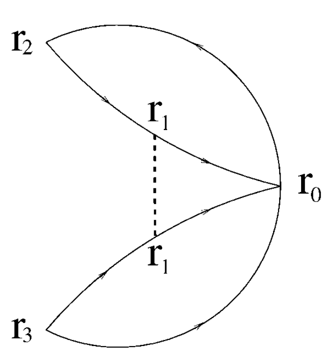

FIG. 3. The vertex.

$$V = \frac{4\pi}{\ell}\left(\frac{\ell}{4\pi}\right)^2 \int d^3r_1 \times \langle G_\omega(\vec{r}_1 - \vec{r}_0)\rangle \langle G^*_\omega(\vec{r}_1 - \vec{r}_0)\rangle f^2_\omega(|\vec{r}_0 - \vec{r}_1|) = \frac{\ell}{4\pi}\int d^3\rho\, \frac{1}{(4\pi\rho)^2}\, e^{-2\rho/\ell}\left(\frac{\sin k_0\rho}{k_0\rho}\right)^2 . \quad (10)$$

The integral is dominated by the region of small ρ, i.e., $\rho \simeq k_0^{-1} \ll \ell$, so that the exponential can be replaced by 1, which leads to $V = \ell/32\pi k_0$. The diagram in Fig. 2 is, thus, equal to

$$\frac{\ell}{32\pi k_0} T(\vec{r}_0, \vec{r}) T(\vec{r}_0, \vec{r} + \Delta\vec{r}) \left(\frac{\ell}{4\pi}\right)^2, \quad (11)$$

where the factor $(\frac{\ell}{4\pi})^2$ accounts for the two vertices connecting the diffusion ladders to the observation points $\vec{r}$ and $\vec{r} + \Delta\vec{r}$. Recalling that $\langle I_\omega(\vec{r})\rangle = (\ell/4\pi)^2 T(\vec{r}, \vec{r}_0)$, and similarly for $\langle I_\omega(\vec{r} + \Delta\vec{r})\rangle$, one can rewrite Eq. (11) as $(\pi/2k_0\ell)\langle I_\omega(\vec{r})\rangle \langle I_\omega(\vec{r} + \Delta\vec{r})\rangle$. Finally, the diagram in Fig. 2 should be assigned a combinatorial factor 2, since one could connect the two external Green's function by a dashed line, instead of connecting the two internal ones as in Fig. 2. (Only lines with arrows in opposite directions, corresponding to a pair of G, G^* need to be connected.) Thus, the contribution of the diagram in Fig. 2 to the normalized correlation function [Eq. (6)] is

$$C_0(\Delta r) = \frac{\pi}{k_0\ell}, \quad (12)$$

where the notation C_0 is used to distinguish this term from the other types of correlations, usually designated [1] as C_1, C_2, and C_3. For the geometry considered, i.e., point source in an infinite three-dimensional medium, the C_0 term is much larger than the other types of long-range correlations. Moreover, it does not decay in space (infinite-range correlation).

It is worth noting that the C_0 term resembles the C_2 term [1,2], in the sense that both terms describe correlation between two distant points, for intensity propagating from a single source. However, C_0 is not contained in C_2. Indeed, C_2 is described by a four-ladder diagram and, thus, cannot contain the diagram in Fig. 2. Moreover, in computing and interpreting the C_2 correlation it is essential that all four ladders are "at work" and can be treated in the diffusion approximation [1,2]. For a point source in three dimensions the C_2 term is of order $(k_0\ell)^{-2}$, which is much smaller than the C_0 term in Eq. (12).

So far $\Delta r \gg \ell$ was assumed. The case $\Delta r = 0$, however, is also of interest. For this case the diagram in Fig. 2 gives a contribution to the second moment of intensity. For the normalized intensity, $\tilde{I}(\vec{r}) \equiv I(\vec{r})/\langle I(\vec{r})\rangle$, this contribution is $2C_0(0) = 2\pi/k_0\ell$ [the extra factor 2, as compared to Eq. (12), appears because for $\Delta r = 0$ one can pair also Green's functions emerging from point $\vec{r}$]. This contribution represents a small correction to the Rayleigh value $\langle \tilde{I}^2\rangle = 2$. The knowledge of the correction enables one to compute deviations of the intensity distribution $P(\tilde{I})$ from the Rayleigh function $P_0(\tilde{I}) = \exp(-\tilde{I})$. The calculation is practically identical to the one performed by Mirlin *et al.* [3], with one important difference: Ref. [3] considered quasi-one-dimensional geometry, where the correction was quite different from the present value $2\pi/k_0\ell$. For the present, three-dimensional case the result is

$$P(\tilde{I}) \simeq \exp\left(-\tilde{I} + \frac{\pi}{2k_0\ell}\tilde{I}^2\right), \qquad \tilde{I} \ll k_0\ell\,. \qquad (13)$$

The asymptotic tail of the distribution, for $\tilde{I} \gg k_0\ell$, can be obtained by the method of optimal fluctuation [4]. For $\tilde{I} \simeq k_0\ell$ the tail should smoothly match expression (13). A detailed calculation of $P(\tilde{I})$ will be presented in a separate publication.

One can easily extend the calculation to include correlations at different frequencies. The object to consider is the same as in Eq. (6), but with $I_{\omega+\Delta\omega}(\vec{r} + \Delta\vec{r})$ instead of $I_\omega(\vec{r} + \Delta\vec{r})$. The only difference in the calculation is that, in Eq. (7) for V, one has to substitute $\omega + \Delta\omega$ for ω in the last three Green's functions. This leads to a replacement $\omega \to \omega + \Delta\omega$ in one of the Green's functions and in one of the f factors in Eq. (10). For $\Delta\omega \ll \omega$ this replacement leads only to an inessential correction to the constant value in Eq. (12).

In conclusion:

(i) It has been shown that the intensity pattern, produced by a wave propagating in a random medium, can exhibit a new type of correlation, designated as C_0. This correlation is of order $1/k_0\ell$ and has infinite range in space and frequency. Unlike the other types of long-range correlations, C_2 and C_3, the C_0 term is dominated by the configuration of the disorder near the source.

(ii) Being sensitive to the short-distance properties of the disorder, the C_0 term cannot be universal; i.e., its magnitude should depend on the specific type of the disorder. For the white-noise Gaussian disorder, considered in this note, C_0 is determined by the parameter $k_0\ell$. However, for Gaussian disorder with a finite correlation length d, or for discrete impurities of size d, the C_0 term should depend on d.

(iii) Let us emphasize that this paper considers a propagating wave, in an open system. A somewhat different problem pertains to the statistics of eigenmodes, and eigenfrequencies, in a closed disordered system. Nonuniversal terms, of the same origin as these considered here, exist also in that case, as has been recently discussed by Mirlin [5].

(iv) The C_0 term is also present in lower dimensionalities. The quasi-one-dimensional geometry (a tube of length L and cross section A) is of particular interest, since it is often used in experiments [6]. The C_2 and C_3 terms in this geometry are of order g^{-1} and g^{-2}, respectively, where $g \simeq k_0^2 A\ell/L$. Thus, for a point source inside the tube, the C_0 term will dominate as long as $k_0\ell \leq g$, i.e., $L < Ak_0$. This implies that our earlier calculation [3], which left out the C_0 term, is valid only for $L > Ak_0$.

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