Mixing Property of Triangular Billiards

Giulio Casati[1] and Tomaž Prosen[2]
[1]*International Center for the Study of Dynamical Systems, Universita'degli Studi dell'Insubria, via Lucini, 3, I-22100 Como, and Istituto Nazionale di Fisica della Materia and INFN, Unità di Milano, Italy*
[2]*Physics Department, Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, 1111 Ljubljana, Slovenia*
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We present numerical evidence which strongly suggests that irrational triangular billiards (all angles irrational with π) are mixing. Since these systems are known to have zero Kolmogorov-Sinai entropy, they may play an important role in understanding the statistical relaxation process.

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After the pioneering paper of Fermi, Pasta, and Ulam [1] and the mathematical works of the Kolmogorov school, the modern ergodic theory can now account for the rich variety of different statistical behaviors of classical dynamical systems. These properties range from complete integrability to deterministic chaos. The relatively few rigorous results so far available have provided a firm guide for a large amount of analytical and numerical work which gave a strong impulse to the field of nonlinear dynamics. Needless to say, several problems remain to be solved. For example, it is known that mixing property guarantees correlations decay and relaxation to statistical equilibrium. It is, however, not known whether or not this property is sufficient for a meaningful statistical description of the relaxation process or whether the strongest property of positive Kolmogorov-Sinai (KS) entropy is required. Even much less clear is the situation in quantum mechanics. For example, to what extent the statistical distribution of energy levels of a quantum conservative Hamiltonian system is related to the different properties in the ergodic hierarchy is an open question.

Examples of classically completely integrable or deterministic random systems have been widely studied in the literature. However, to our knowledge, there are no physical examples of systems which possess the mixing property only (with zero KS entropy). In this respect, the best candidates are billiards in 2D triangles but, in spite of 30 years of investigations, no definite statement can be made concerning their dynamical properties [2]. Moreover [3], "a prevailing opinion in the mathematical community is that polygonal billiards are never mixing, but this has not been established."

In this paper, we provide strong numerical evidence that generic triangular billiards, with *all angles irrational* with π, are mixing. This result can play an important role in the understanding of nonequilibrium statistical mechanics since it fills a gap in the ergodic hierarchy. Moreover, since the local dynamical instability in these systems is only linear, they have zero algorithmic complexity. This means that, even though it may be very difficult, their analytical solution is not impossible in principle and this may prove to be very important for the future development of nonlinear dynamics.

We consider the motion of a point particle, with unit velocity, inside a generic triangular billiard (with all angles irrational with π, in general). Let us introduce Cartesian coordinates (x, y), and align one side of the triangle along the x axis ($y = 0$), starting at the origin $x = 0$, and let us choose its length to be unity. We denote the angle at the origin $(0, 0)$ by α and the angle at $(1, 0)$ by β. Hence the other two sides of the triangle are given by the equations $y = (\tan\alpha)x$, and $y = (\tan\beta)(1 - x)$.

We would like to draw attention to the following fact: the dynamics of a point particle in a triangular billiard is *equivalent* to the motion of three particles on a ring with different masses, m_1, m_2, and m_3. Let the ring have circumference 1. The motion of 3-particles 1D gas with coordinates q_1, q_2, q_3 ($q_1 \le q_2 \le q_3 \le q_1 + 1$) is governed by the Hamiltonian $H = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2 + \frac{1}{2}m_3\dot{q}_3^2$, with elastic collisions at $q_1 = q_2$, $q_2 = q_3$, and $q_3 = q_1 + 1$. Introducing the notation

$$\vec{r} = (\sqrt{m_1}q_1, \sqrt{m_2}q_2, \sqrt{m_3}q_3)$$

and the orthogonal transformation

$$x = \frac{1}{\sqrt{(m_1 + m_2)M}}(-\sqrt{m_1m_3}, -\sqrt{m_2m_3}, m_1 + m_2) \cdot \vec{r},$$
$$y = \frac{1}{\sqrt{m_1 + m_2}}(-\sqrt{m_2}, \sqrt{m_1}, 0) \cdot \vec{r},$$
$$z = \frac{1}{\sqrt{M}}(\sqrt{m_1}, \sqrt{m_2}, \sqrt{m_3}) \cdot \vec{r},$$

with $M = m_1 + m_2 + m_3$, the Hamiltonian writes: $H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ where, as can be checked by straightforward calculations, the motion in the (x, y) plane is bounded by specular reflections from the sides of the triangle with angles

$$\tan\alpha = \sqrt{\frac{m_2M}{m_1m_3}}, \quad \tan\beta = \sqrt{\frac{m_1M}{m_3m_2}}, \quad \tan\gamma = \sqrt{\frac{m_3M}{m_2m_1}}. \tag{1}$$

The motion in the z coordinate trivially separates and corresponds to the center-of-mass motion in the 3-particles 1D gas.

Notice, however, that triangular billiards with one angle larger than $\pi/2$ cannot be related to the 3-particles 1D gas, since from the above formula it follows that $\alpha, \beta, \gamma < \pi/2$. A particular case is given by the isosceles triangle which is dynamically equivalent to the right triangle billiard, with, e.g., $\gamma = \pi/2$. The latter corresponds to $m_3 = \infty$, i.e., to the 1D motion of two particles with masses m_1, m_2 between hard walls [4].

The dynamics of triangular billiards can be divided into three classes:

(A) *All* angles rational with π: The dynamics of such triangles is not ergodic; in fact, it is *pseudointegrable*, i.e., it possesses 2D invariant surfaces of high genus in 4D phase space (see [3] and Refs. therein).

(B) *Only one* angle rational with π: Such are generic right triangles, for example. In recent numerical experiments evidence has been given that right, irrational, triangular billiards are ergodic and weakly mixing [5].

(C) *All* angles *irrational* with π: Surprisingly, to the best of our knowledge, this generic class of triangles has been somehow overlooked in previous numerical studies and will be the main object of the present paper. It is within this class that one may now hopefully look for ergodic and mixing behavior.

From the rigorous point of view not much is known beyond the fact that the set of ergodic triangles is dense in a suitable topology. We recall that a dynamical system T^t: $\vec{x}(0) \to \vec{x}(t)$ with invariant measure $\mu(\vec{x})$ in phase space $\mathcal{M}$ is mixing if, for any L^2 pair of observables in phase space, their time correlation function asymptotically vanishes

$$\lim_{t\to\infty}\left[\int_{\mathcal{M}} d\mu f(T^t\vec{x})g(\vec{x}) - \int_{\mathcal{M}} d\mu f(\vec{x})\int_{\mathcal{M}} d\mu g(\vec{x})\right] = 0\,.$$

The map T^t may represent a continuous flow (t real) or a discrete map (t integer). In this paper we shall mainly consider the dynamics given by a discrete Poincaré map which corresponds to the collisions of the orbits with the horizontal side $y = 0$. The reduced phase space—surface of section (SOS)—is a rectangle, parametrized by the coordinate $0 \le x \le 1$ and by the corresponding canonical momentum $-1 \le p_x = \sin\vartheta \le 1$ (ϑ = angle of incidence), with the invariant measure $d\mu = dx dp_x$.

As a first step we have performed an accurate check of ergodicity. We have done this in two independent ways:

(i) By dividing the phase space (SOS) in a large number $N = N_1 \times N_1$ of cells and then computing the number $n(t)$ of cells which are visited by a single orbit up to discrete time t [6]. Then we computed the average relative measure $r(t) = \langle n(t)/N\rangle$ of visited phase space up to time t where $\langle\cdot\rangle$ denotes phase space average over sufficiently many randomly chosen initial conditions. As it is known, for the so-called *random model* of completely uniform (ergodic) and random dynamics (obtained by assuming that, at each discrete time step, a point has probability $1/N$ to fall in any of the N cells), one can derive the simple scaling law [6]

$$r(t) = r_{\mathrm{RM}}(t) = 1 - \exp(-t/N)\,. \tag{2}$$

Expression (2) (for arbitrary but sufficiently fine mesh N) should be considered as a sufficient but not necessary condition for ergodicity. In fact, the dynamics would obey the law (2) only when the system does not possess any nontrivial time scale. In Fig. 1 we show the results of numerical computations for triangles of class (B) and (C). We take, for example, a right triangle B with angles $\alpha = (\sqrt{5} - 1)\pi/4$, $\beta = \pi/2 - \alpha$, $\gamma = \pi/2$ and a generic triangle C with angles $\alpha = (\sqrt{2} - 1)\pi/2$, $\beta = (\sqrt{5} - 1)\pi/4$, $\gamma = \pi - \alpha - \beta$. It is seen that there is a drastic difference between the two cases: the generic triangle C excellently follows the law (2), whereas the right triangle B, even though ergodic [5], strongly deviates and explores the phase space extremely slowly, with a hierarchy of long time scales related to a strong sticking of the orbit in momentum (p_x) space.

(ii) By comparing the *time averaged* correlation function for a single, but very long orbit, $C_a^t(t) = \lim_{T\to\infty}\frac{1}{T}\sum_{n=1}^{T} a(n)a(n + t)$ with the phase averaged correlation function $C_a^p(t) = \langle a(0)a(t)\rangle$ which is computed by Monte Carlo averaging over many short orbits with different randomized initial conditions. In Fig. 2 we plot the difference between the two curves for the generic triangle C which turns out to be of the same order as the statistical error.

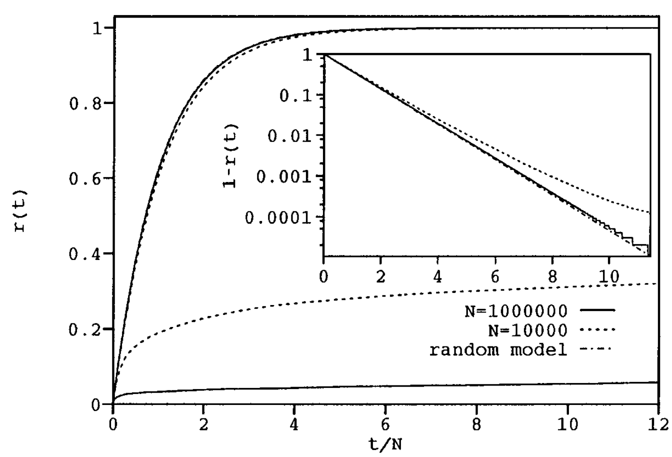

FIG. 1. The relative measure $r(t)$ of the visited phase space as a function of discrete time t. The full and dashed curves refer to different discretizations of phase space, namely to $N = 10^6$ and $N = 10^4$ cells, respectively. The two upper curves refer to the generic triangle C, whereas the two lower curves refer to the golden right triangle B (see text for details). The random model, Eq. (2), is given by the dash-dotted curve which is almost indistinguishable from the numerical curve for the triangle C at $N = 10^6$. In order to better display the excellent agreement with Eq. (2), we plot in the inset the function $1 - r(t)$ for the generic triangle C in the semilog scale. At $N = 10^6$, and $N = 10^4$, averages over 200, and 4000 orbits with randomized initial conditions have been used, respectively.

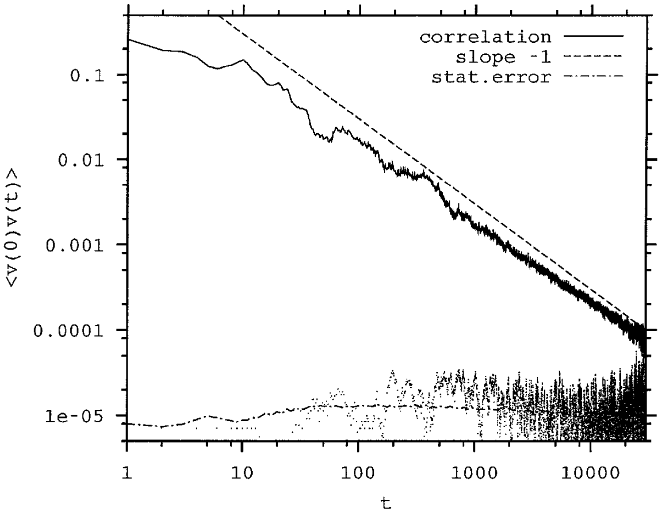

FIG. 2. The velocity autocorrelation function $C^p_{p_x}(t)$ in log-log scale for the generic triangle C (full curve), which is obtained as the average over 1.2×10^6 different orbits of length 32 768 (full curve). The dashed line gives the slope -1.00. The dash-dotted curve gives the estimated statistical error of the correlation function, while the scattered dots around it mark the difference between the phase averaged and the time averaged correlation $|C^p_{p_x}(t) - C^t_{p_x}(t)|$ where the time average is computed from a single orbit of length $T = 32\,768 \times 1.2 \times 10^6$ with initial condition $x_0 = 0.23\,456$, $p_{x0} = 0.34\,567$. The statistical error is estimated as the standard deviation of a sequence of $M = 1000$ partial averages of length T/M each.

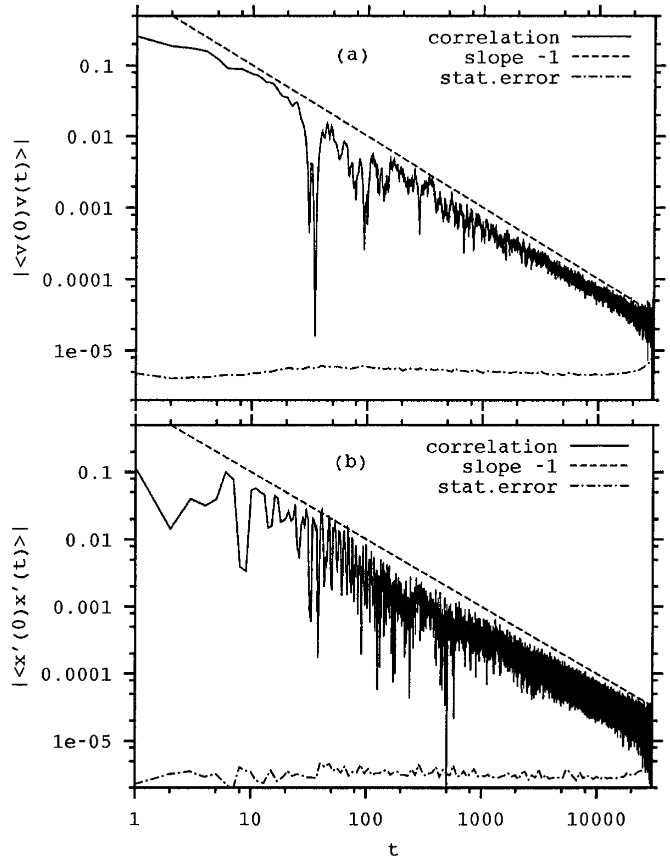

FIG. 3. The absolute values of velocity (a) and position (b) correlation function, $|C^t_p(t)|$ and $|C^t_{x'}(t)|$, for an orbit of length 10^{11} in the generic triangular billiard D. Initial condition x_0, p_{x0} and line coding are the same as in Fig. 2.

The above numerical results are surprisingly much clearer than expected; they demonstrate ergodic behavior and they make it reasonable to expect mixing behavior also. We now turn our attention to the latter question.

We have performed extensive numerical computations of autocorrelation functions of different observables, namely, momentum (horizontal component of velocity) $v = p_x$, symmetrized position $x' = 2x - 1$, and also characteristic functions of sets in phase space. In all cases, we have found clear numerical evidence of *power-law decay* of correlation functions over about 4 orders of magnitude. For a given triangle, correlation functions of different observables decay with the same empirical exponent which is typically very close to -1. In Fig. 2 we show the velocity correlation function of the triangle C, which decays with exponent with the fitted value $-\sigma = -0.90 \pm 0.02$. However, since the triangle C is not too far from the right triangle, we have chosen another, similar generic triangle D with the irrational angles $\alpha = (\sqrt{2} - 1)\pi/2$, $\beta = 1$, $\gamma = \pi - \alpha - \beta$ and show in Fig. 3 its velocity and position correlation function which decay with empirical exponent $-\sigma = -0.94 \pm 0.04$. We have performed extensive numerical calculations of correlation functions for many different generic triangles, and the exponent of decay has always been very close to -1. There are even cases where the fitted decay exponent seems to be slightly below -1 ($\sigma > 1$); however, in such cases, correlation functions appear much more noisy, and it is difficult to make precise statements.

Finally, we have performed a different and powerful statistical test, namely we have computed the Poincaré recurrences or the return time statistics, i.e., the probability $P(t)$ for an orbit not to stay outside a given subset $\mathcal{A} \subset \mathcal{M}$ for a time longer than t. It has been conjectured [7,8] (see also [9] for accurate definition and further references) that the integrated probability $P_i(t) = \sum_{t'=t}^{\infty} P(t')$ should be intimately connected to the correlation decay. More precisely, if the correlation function decays as a power law with exponent $-\sigma$, then the integrated return probability should decay with a similar exponent $P(t) \sim t^{-\mu-1}, P_i(t) \sim t^{-\mu}$, where $\mu = \sigma$. In Fig. 4 we show the recurrence probability $P(t)$ and the integrated probability $P_i(t)$ computed with respect to half space $\mathcal{A} = \{(x, p_x); p_x > 0\}$ for the same data as in Fig. 3 (triangular billiard D). We have found that numerical results are perfectly consistent with $\mu = \sigma = 1$, i.e., with $1/t^2$ decay of return probability $P(t)$.

To summarize, the numerical obtained value of exponent σ is close to -1, but in some cases is slightly less and in other cases slightly larger. We attribute this fact to long transient times which are caused by the intricate

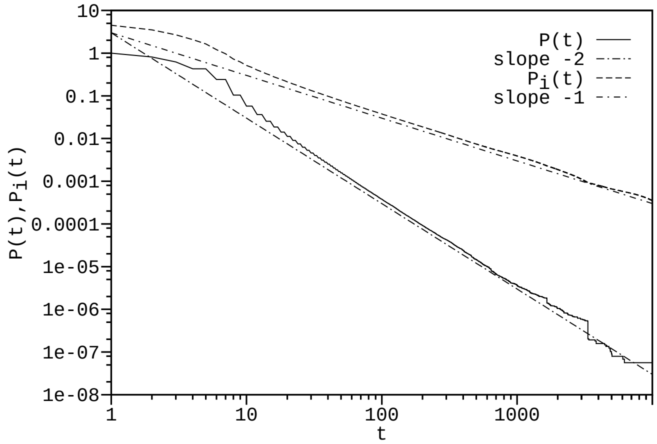

FIG. 4. The return probability $P(t)$ (full curve), and the integrated return probability $P_i(t)$ (dashed curve), for the same orbit (triangle D) as in Fig. 3. The dash-dotted lines have slopes -2 and -1, respectively.

interplay of arithmetic properties of angles α, β, γ and by the presence of periodic orbits. In this connection, we would like to mention that we have numerically investigated the existence of periodic orbits, and we have found that their number increases very slowly with their period. The number of nonequivalent periodic orbits with lengths up to l (collisions with the boundary) typically increases *slower* than l.

A qualitative argument which leads to the $1/t$ decay of correlations reads as follows: let us consider a fixed, sufficiently small, region $\mathcal{A}$ in phase space around a periodic orbit and study the l iterates of the Poincaré map, where l is the period of the orbit. Because of the linear (in)stability, the orbit which starts in a small square a_ϵ of side ϵ centered on the periodic orbit and contained in $\mathcal{A}$, will remain in $\mathcal{A}$ for a time $t \geq 1/\epsilon$. It follows that the probability $P(1/\epsilon)$ for an orbit to stay in the region $\mathcal{A}$ for a time $t \geq 1/\epsilon$ is proportional to the probability of the orbit to visit the square a_ϵ. Now, the linear instability implies that an orbit which enters the square a_ϵ will remain inside a_ϵ for a constant time which does not depend on ϵ as $\epsilon \rightarrow 0$. As a consequence, from the Liouville theorem it follows that the probability for an orbit to visit the square a_ϵ will be proportional to its area ϵ^2. This is the probability $P(1/\epsilon)$ to remain in the region $\mathcal{A}$ for a time $t \geq 1/\epsilon$. This leads to the relation $P(t) \propto 1/t^2$ as confirmed by numerical results.

Since the question discussed in this paper is a very delicate one, we have put particular attention to the accuracy of our numerical computations. Among the several different tests, we have developed two independent computer codes, one based on the billiards dynamics in a 2D plane, and the other based on the three particles 1D gas dynamics: the results agree. Moreover, since the instability here is only linear in time, and the machine precision is $\sim 10^{-16}$, the numerical errors are always below the statistical errors.

The central question is to what extent the results presented here can be considered as a definite or convincing evidence for the mixing property of generic triangular billiards. The correlation decay of some particular class of functions is certainly compatible with a weaker property than mixing. However, in this paper we have compared the correlations decay of different variables (velocity, position, characteristic functions of phase space sets, etc). We have also considered the dynamics given by the discrete Poincaré map relative to different sides of the same triangle: all these correlations exhibit a power law decay with exponent very close to -1, and make very plausible the conjecture that correlation functions in generic triangles decay as $\propto 1/t$. Of particular significance is the behavior of the return probability $P(t)$ which decays with the expected power law $\propto 1/t^2$. All the above results (including those described in Fig. 1) lead to the conclusion that, outside any reasonable doubt, generic irrational triangles are mixing. The particular case of isosceles or right triangles is much less clear. The behavior of correlations in such a case is noisy and the exponent of the decay is too small to allow for any definite conclusion.

The fact that generic triangles have zero Kolmogorov-Sinai entropy and yet they have a very nice mixing behavior without any time scale, may prove to be very useful for understanding the dynamical basis of the relaxation process to the statistical equilibrium. Moreover, the analysis of their quantum behavior will contribute to the current efforts for the construction of a statistical theory of quantum dynamical systems.

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