An Alternative to Compactification

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Conventional wisdom states that Newton's force law implies only four noncompact dimensions. We demonstrate that this is not necessarily true in the presence of a nonfactorizable background geometry. The specific example we study is a single 3-brane embedded in five dimensions. We show that even without a gap in the Kaluza-Klein spectrum, four-dimensional Newtonian and general relativistic gravity is reproduced to more than adequate precision.

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There exists evidence that convinces us that we live in four noncompact dimensions. Certainly standard model matter cannot propagate a large distance in extra dimensions without conflict with observations. As has recently been emphasized, this can be avoided if the standard model is confined to a (3 + 1)-dimensional subspace, or "3-brane," in the higher dimensions [1,2]. However, this solution will not work for gravity, which necessarily propagates in all dimensions as it is the dynamics of spacetime itself. The experimental success of Newton's $1/r^2$ law and general relativity then seems to imply precisely four noncompact dimensions. Additional dimensions are nonetheless acceptable, but they should be compact and smaller than a millimeter so that they would not have been resolved in our shortest distance tests of gravity. One further piece of evidence is that if there are n extra compact dimensions, the Planck scale is related to the higher-dimensional scale of gravity, M, through the relation $M_{\rm Pl}^2 = M^{2n} V_n$, where V_n is the extra-dimensional volume.

The point of this Letter is to argue that none of the statements about gravity in the previous paragraph is necessarily true. The previous conclusions rely on a factorizable geometry, namely, the metric of the four familiar dimensions is independent of position in the extra dimensions. The story can change significantly when this assumption is violated. Perhaps the most dramatic consequence is that we can live in $4 + n$ *noncompact* dimensions, in perfect compatibility with experimental gravity. We will give an example where $n = 1$. We will show that $M_{\rm Pl}$ is determined by the higher-dimensional curvature rather than the size of the extra dimension. This curvature is not in conflict with four-dimensional Poincaré invariance. Earlier work on noncompact extra dimensions focused on trapping matter [3] or on finite-volume dimensions [4].

The reason the above statements can be true is that a curved background can support a "bound state" of the higher-dimensional graviton, which is localized in the extra dimensions. This can be understood as follows. Small gravitational fluctutations satisfy a wave equation of the form

$$[\partial_\mu \partial^\mu - d_j d^j + V(z_j)]\hat{h}(x^\mu, z_j) = 0\,, \tag{1}$$

with a nontrivial "potential," V, arising from the curvature. (We have dropped Lorentz indices on the fluctuations here for simplicity.) General fluctuations can be written as superpositions of eigenmodes, $\hat{h} = e^{ip\cdot x}\hat{\psi}(z)$, where

$$[-d_j d^j + V(z)]\hat{\psi}(z) = -m^2 \hat{\psi}(z)\,, \tag{2}$$

and $p^2 = m^2$. This implements the Kaluza-Klein (KK) reduction of the higher-dimensional gravitational fluctuations in terms of four-dimensional KK states, with the mass squared m^2 given by the eigenvalues of Eq. (2), and with fixed wave function in the extra dimension, $\hat{\psi}(z)$. It is useful to note that Eq. (2) takes the form of an analog nonrelativistic quantum mechanics problem.

If there is a zero mode (which is guaranteed if the background preserves four-dimensional Poincaré invariance) that is also a normalizable state in the spectrum of Eq. (2), it corresponds to a four-dimensional graviton. In addition there exists a tower of higher KK modes. If there were a gap, as is conventional in product space compactifications, one reproduces four-dimensional gravity up to the scale determined by the gap. Instead, in our example because of a nontrivial potential we find a very interesting situation where there is a single bound state of the analog quantum mechanics problem corresponding to a massless four-dimensional graviton, and whose extra-dimensional wave function is centered on a 3-brane to which the standard model is confined. There is also a continuum KK spectrum with no gap. We nonetheless reproduce Newtonian gravity and other four-dimensional general relativistic predictions at low energy and long distance. The example we give will be an effective four-dimensional theory in five noncompact dimensions.

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The setup for our theory is a single 3-brane with a positive tension, embedded in a five-dimensional bulk spacetime. In order to carefully quantize the system and treat the non-normalizable modes which will appear in the Kaluza-Klein reduction, we choose to first work in a finite volume by introducing another brane at a distance πr_c from the brane of interest, and taking the branes to be the boundaries of a finite fifth dimension. We will eventually take this second brane to infinity, thereby removing it from the physical setup. The associated action is

$$S = S_{\text{gravity}} + S_{\text{brane}} + S_{\text{brane}'},$$

$$S_{\text{gravity}} = \int d^4x \int dy \sqrt{-G}\{-\Lambda + 2M^3 R\}, \tag{3}$$

$$S_{\text{brane}} = \int d^4x \sqrt{-g_{\text{brane}}}\{V_{\text{brane}} + \mathcal{L}_{\text{brane}}\},$$

where R is the five-dimensional Ricci scalar made out of the five-dimensional metric, G_{MN}. The coupling to the branes and their fields and the related orbifold boundary conditions are described in Ref. [2]. (The new coordinate y is $r_c\phi$ in the coordinates of Ref. [2].)

The solution to Einstein's equations was derived in Ref. [2] and we quote it here:

$$ds^2 = e^{-2k|y|}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2, \tag{4}$$

where $0 \le y \le \pi r_c$ is the extra-dimensional coordinate and r_c is essentially a compactification "radius." This is just a slice of the symmetric space, AdS_5. The solution holds only when the boundary and bulk cosmological terms are related by

$$V_{\text{brane}} = -V_{\text{brane}'} = 24M^3k, \qquad \Lambda = -24M^3k^2, \tag{5}$$

which we hereby take to be the case. Notice that in the solution here, we have reversed the roles of the "visible" and "hidden" branes relative to Ref. [2]. Whereas in the solution to the hierarchy problem proposed in Ref. [2] the massless graviton wave function is biggest on the hidden brane, in the scenario considered here it is critical that the graviton is "bound" to the visible brane.

We now remind the reader of the derivation of the four-dimensional effective Planck scale, M_{Pl}. The four-dimensional graviton zero mode follows from our solution, Eq. (4), by replacing the Minkowski metric by a four-dimensional metric, $\overline{g}_{\mu\nu}(x)$. It is described by an effective action following from substitution into Eq. (3),

$$S_{\text{eff}} \supset \int d^4x \int_0^{\pi r_c} dy\, 2M^3 r_c e^{-2k|y|}\overline{R}, \tag{6}$$

where $\overline{R}$ denotes the four-dimensional Ricci scalar made out of $\overline{g}_{\mu\nu}(x)$, in contrast to the five-dimensional Ricci scalar, R, made out of $G_{MN}(x,y)$. Because the effective field is four-dimensional, we can explicitly perform the y integral to obtain a purely four-dimensional action. From this we derive

$$M_{\text{Pl}}^2 = 2M^3 \int_0^{\pi r_c} dy\, e^{-2k|y|} = \frac{M^3}{k}[1 - e^{-2kr_c\pi}]. \tag{7}$$

We see that there is a well-defined value for M_{Pl}, even in the $r_c \to \infty$ limit. This is a clue that one can get a sensible effective four-dimensional theory, with the usual Newtonian force law, even in the infinite radius limit, and provides a sharp contrast to the product-space expectation that $M_{\text{Pl}}^2 = M^3 r_c \pi$.

Clearly, there is no problem with taking the $r_c \to \infty$ limit of the background metric given above. This will remove the "regulator" brane from the setup. However, we still need to determine whether the spectrum of *general* linearized fluctuations $G_{MN} = e^{-2k|y|}\eta_{\mu\nu} + h_{\mu\nu}(x,y)$ is consistent with four-dimensional experimental gravity. This requires an understanding of all modes that appear in the assumed four-dimensional effective theory. We therefore perform a Kaluza-Klein reduction down to four dimensions. To do this, we need to do a separation of variables; we write $h(x,y) = \psi(y)e^{ip\cdot x}$, where $p^2 = m^2$ and m^2 permits a solution to the linearized equation of motion for tensor fluctuations following from Eq. (3) expanded about Eq. (4):

$$\left[\frac{-m^2}{2}e^{2k|y|} - \frac{1}{2}\partial_y^2 - 2k\delta(y) + 2k^2\right]\psi(y) = 0, \tag{8}$$

where our boundary conditions tell us to consider only even functions of y, describing the infinite half-line. The effect of the regulator brane will be considered later; here it has been taken to infinity. The $\mu\nu$ indices are the same in all terms if we work in the gauge where $\partial^\mu h_{\mu\nu} = h_\mu^\mu = 0$, so they are omitted. Here m is the mass of the KK excitation.

It is more convenient to put the above equation into the form of an analog nonrelativistic quantum mechanics problem by making a change of variables, $z \equiv sgn(y) \times (e^{k|y|} - 1)/k$, $\hat{\psi}(z) \equiv \psi(y)e^{k|y|/2}$, $\hat{h}(x,z) \equiv h(x,y) \times e^{k|y|/2}$. Equation (8) then reads

$$\left[-\frac{1}{2}\partial_z^2 + V(z)\right]\hat{\psi}(z) = m^2\hat{\psi}, \tag{9}$$

where

$$V(z) = \frac{15k^2}{8(k|z| + 1)^2} - \frac{3k}{2}\delta(z). \tag{10}$$

Much can be understood from the general shape of this analog nonrelativistic potential.

First, the δ function supports a single normalizable bound state mode; the remaining eigenstates are continuum modes. Furthermore, since the potential falls off to zero as $|z| \to \infty$, there is no gap, and the continuum modes asymptote to plane waves. These plane waves decay subasymptotically, corresponding to their tunneling through the potential to get to $z = 0$. By tuning of the cosmological terms we have ensured that the bound state mode corresponds to a massless four-dimensional graviton, $m = 0$. In the four-dimensional description following Kaluza-Klein reduction, we have tuned the effective cosmological constant to zero. The continuum KK states have all possible $m^2 > 0$.

The precise continuum modes are given in terms of Bessel functions and are a linear combination of $(|z| + 1/k)^{1/2} Y_2(m(|z| + 1/k))$ and $(|z| + 1/k)^{1/2} J_2(m(|z| + 1/k))$. The zero mode wave function follows (after changing variables) from Eq. (4), $\hat{\psi}_0(z) = k^{-1}(k|z| + 1)^{-3/2}$. [Though the zero mode is not a Bessel function, it is the limit of $m^2(|z| + 1)^{1/2} Y_2(m(|z| + 1))$ when $m \to 0$.] We can better understand the KK modes by studying the small and large argument limits of the Bessel functions. For small $\frac{m}{k}(|z| + 1)$ we have

$$J_2(m(|z| + 1/k)) \sim \frac{m^2(|z| + 1/k)^2}{8}, \qquad Y_2(m(|z| + 1/k)) \sim -\frac{4}{\pi m^2(|z| + 1/k)^2} - \frac{1}{\pi}. \tag{11}$$

Therefore to satisfy the boundary condition implied by the δ-function potential on the brane at $z = 0$, for small m (relevant at long distances) we must choose the linear combination,

$$\hat{\psi}_m \sim N_m (|z| + 1/k)^{1/2} \left[Y_2(m(|z| + 1/k)) + \frac{4k^2}{\pi m^2} J_2(m(|z| + 1/k)) \right]. \tag{12}$$

Here N_m is a normalization constant. For large mz,

$$\sqrt{z}\, J_2(mz) \sim \sqrt{2\frac{2}{\pi m}} \cos\left(mz - \frac{5}{4}\pi\right), \qquad \sqrt{z}\, Y_2(mz) \sim \sqrt{\frac{2}{\pi m}} \sin\left(mz - \frac{5}{4}\pi\right). \tag{13}$$

Let us now consider what happens when we reintroduce the regulator brane at $y_c \equiv \pi r_c$, that is, $z_c \equiv (e^{k\pi r_c} - 1)/k$. It simply corresponds to a new boundary condition at z_c,

$$\partial_z \hat{\psi}(z_c) = -\frac{3k^2}{2(kz_c + 1)} \hat{\psi}(z_c). \tag{14}$$

It is easy to check that our zero mode satisfies this new condition. The KK excitations are now quantized by this condition, however. For large z_c they are all in the plane-wave asymptotic regime of Eq. (13) when they satisfy the new condition. Therefore their masses are approximately quantized in units of $1/z_c$. Furthermore, their normalization constants are predominantly those of plane waves, in particular, $N_m \sim \pi m^{3/2}/(4k^2\sqrt{z_c})$.

Having obtained the large but finite r_c asymptotics we can determine the proper measure for sums over the continuum states in the $r_c \to \infty$ limit. Because these asymptotics were dominated by plane-wave behavior, this measure is simply dm/k after dropping the $1/\sqrt{z_c}$ factor in N_m to go to a continuum normalization. We have also demonstrated the claim made in Ref. [2] that when z_c is kept large but finite, the KK states are quantized in units of $1/z_c$, which in that paper corresponded to the TeV scale. Also note that the normalized KK wave functions at the brane at z_c are all of order $1/\sqrt{z_c}$ since they are all plane waves at a maximum or minimum according to Eq. (14), which is kz_c times larger than $\hat{\psi}_0(z_c)$. This proves the claim of Ref. [2] that the KK states couple 10^{15} more strongly to matter on the brane at z_c than does the massless graviton.

By taking $r_c \to \infty$ we have obtained a semi-infinite extra dimension. It is trivial to extend this to a fully infinite extra dimension by simply allowing even and odd functions of z rather than the restriction to purely even functions demanded by the orbifold conditions. From now on we will consider this to be the case.

We can now compute the effective nonrelativistic gravitational potential between two particles of mass m_1 and m_2 on our brane at $z = 0$, that is, the static potential generated by exchange of the zero-mode and continuum Kaluza-Klein mode propagators. It is

$$V(r) \sim G_N \frac{m_1 m_2}{r} + \int_0^\infty \frac{dm}{k} G_N \frac{m_1 m_2 e^{-mr}}{r} \frac{m}{k}. \tag{15}$$

Note there is a Yukawa exponential suppression in the massive Green's functions for $m > 1/r$, and the extra power of m/k arises from the suppression of continuum wave functions at $z = 0$ following from Eq. (11), due to the analog tunneling effect discussed above. Therefore, the potential behaves as

$$V(r) = G_N \frac{m_1 m_2}{r}\left(1 + \frac{1}{r^2 k^2}\right). \tag{16}$$

This is why our theory produces an effective four-dimensional theory of gravity. The leading term due to the bound state mode is the usual Newtonian potential; the KK modes generate an extremely suppressed correction term, for k of order the fundamental Planck scale and r of the size tested with gravity. Furthermore, since our propagators are relativistic in general, going beyond the nonrelativistic approximation we find all the proper relativistic corrections, again with negligible corrections from the continuum modes.

Gravitational radiation is described by "cutting" our relativistic propagators. From the small m limit, we also learn that the production of the continuum modes from the brane at $z = 0$ is suppressed by $(dm/k)(m/k)$ due to the continuum wave function suppression there. This is very important, because it means the amplitude to produce the continuum modes in low-energy processes on the brane is extremely small. Were this not the case, we would be continuously losing energy to the additional dimension. Because of this suppression factor, the probability of producing KK modes is suppressed by $(p/k)^2$ relative to the zero mode, where p is the momentum of a process. For k of order the Planck scale, this is extraordinarily small for any process we presently observe, or are ever likely to observe.

So far, we have shown that a scenario with an infinite fifth dimension in the presence of a brane can generate a theory of gravity which mimics purely four-dimensional gravity, both with respect to the classical gravitational potential and with respect to gravitational radiation. However, we have yet to consider the gravitational self-couplings. This is important, because these couplings have been tested at the 10^{-3} level of precision. However, the constraint is really on the graviton coupling to matter fields with *gravitational* strength. Because the KK modes have p/k-*suppressed* coupling to matter on the brane, relative to the zero mode, they are negligible. The zero mode exchanges and self-couplings are just those of a four-dimensional general relativistic dynamics described by Eq. (6).

However, it is important to verify that the energy loss induced by gravitational self-interactions is also insignificant, that is, the coupling of the zero mode to KK modes which do *not* ultimately couple back to matter on the brane. By expanding the gravitation action, it can be seen that for any finite energy, the graviton self-coupling gets large at an energy-dependent value of the coordinate z. However, fluctuations originating on the brane in low-energy processes have only a small probability to get to this large z. Graviton emission and the associated missing energy can be bounded within the framework of our low-energy effective theory and can be shown to be small.

To conclude, we have found that we can consistently exist with an infinite fifth dimension, without violating known tests of gravity. The scenario consists of a single 3-brane in (a piece of) AdS_5 in the bulk and an appropriately tuned tension on the brane. The need for this delicate adjustment is the equivalent of the cosmological constant problem in this context and is taken as a given and not solved.

In this setup, we have found that an inevitable consequence is a bound state graviton mode, whose shape is determined by the brane tension and bulk cosmological constant. There are no very large or small numbers assumed for the different mass scales in the problem, so the four-dimensional Planck scale is comparable to the fundamental mass scale of the higher-dimensional theory. In addition to the bound state mode, there is a continuum of Kaluza-Klein modes. These have very weak coupling to low-energy states on the brane, but are essential to the consistency of the full theory of gravity and would couple strongly to Planck-energy brane processes.

Notice that one interpretation of our result is as a solution to the moduli problem, for the particular modulus determining the distance between two branes. It says that the usual disasterous possibility, namely, that the modulus runs away to infinity, is perfectly acceptable. Furthermore, in the $r_c \to \infty$ limit, the modulus is not coupled to matter on the brane, and the need for a modulus mass is eliminated. It is an interesting question whether one can eliminate further moduli by not compactifying, and whether geometric compactification from a higher-dimensional setup is essential.

Our effective theory is clearly very different from truly compactified theories. The low-dimensional Planck scale and all physical parameters of the effective four-dimensional theory are independent of r_c, so long as it is much greater than $1/k$. At sufficiently low energies, the probability of losing energy to the KK states is very small. From these perspectives, the theory provides a well-defined alternative to geometric compactification. Many interesting questions remain to be addressed; perhaps this new setup can help resolve some unanswered questions in conventional and quantum gravity and cosmology.

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