Scaling Approach to Calculate Critical Exponents in Anomalous Surface Roughening

Juan M. López*

Department of Mathematics, Imperial College, London SW7 2BZ, United Kingdom
(Received 27 July 1999)

We study surface growth models exhibiting anomalous scaling of the local surface fluctuations. An analytical approach to determine the local scaling exponents of continuum growth models is proposed. The method allows one to predict when a particular growth model will have anomalous properties ($\alpha \neq \alpha_{\mathrm{loc}}$) and to calculate the local exponents. Several continuum growth equations are examined as examples.

Kinetic roughening of surfaces in nonequilibrium conditions has been a subject of great interest in the past few years. This is mainly due to the many important applications of the theory of surface growth including molecular-beam epitaxy (MBE), fluid flow in porous media, and fracture cracks among others [1]. Growth processes have very often been shown to exhibit scaling properties that allow one to divide the models into universality classes characterized by the value of the critical exponents [1,2]. The *global* interface width in a system of total lateral size L, which is the root mean square of the fluctuations of the surface height, scales according to the Family-Vicsek ansatz [3] as

$$W(L, t) = t^{\alpha/z} f(L/t^{1/z}), \qquad (1)$$

where the scaling function $f(u)$ behaves as

$$f(u) \sim \begin{cases} u^{\alpha} & \text{if } u \ll 1, \\ \text{const} & \text{if } u \gg 1. \end{cases} \qquad (2)$$

The roughness exponent α and the dynamic exponent z characterize the universality class of the model under study. The ratio $\beta = \alpha/z$ is the time exponent.

However, very recent numerical studies have revealed a rich variety of interesting phenomena, which are poorly understood at present. In particular, it has been shown that many growth models [4–11] exhibit *anomalous* roughening, i.e., a different scaling for the global and the local surface fluctuations. This leads to the existence of an independent *local* roughness exponent α_{loc} that characterizes the local interface fluctuations on scales $l \ll L$ and differs from α.

A general analytical method to determine anomalous exponents from the continuum growth equations is still lacking. In particular, a dynamic renormalization-group (RG) calculation in the Lai, Das Sarma, and Villain (LDV) nonlinear continuum model has had only limited success [12] because of the existence of nonperturbative infrared singularities that are inaccessible to the usual dynamical RG analysis. Therefore, most efforts have been focused on determining the anomalous critical exponents using simulation of discrete models or direct numerical solutions of the Langevin-type equations of growth.

In this Letter, we propose a new analytical approach to determine the scaling exponents of the local surface fluctuations in continuum growth models exhibiting anomalous kinetic roughening. Our method allows one to predict when a particular growth model is expected to have anomalous properties ($\alpha \neq \alpha_{\mathrm{loc}}$) and, in principle, also to determine the local exponents. We illustrate the method by studying several growth equations with and without anomalous scaling.

We shall be interested here in continuum growth models in $d + 1$ dimensions which dynamics is expected to be described by the Langevin-type equation

$$\frac{\partial h}{\partial t} = \Phi(\nabla h) + \eta(\mathbf{x}, t), \qquad (3)$$

where $h(\mathbf{x}, t)$ is the height of the interface at substrate position $\mathbf{x}$ at time t. The actual form of the function $\Phi(\nabla h)$ defines a particular model [13]. $\eta(\mathbf{x}, t)$ is a noise term uncorrelated in space and time, $\langle \eta(\mathbf{x}, t) \eta(\mathbf{x}', t') \rangle = 2D\delta(\mathbf{x} - \mathbf{x}')\delta(t - t')$.

The scaling properties of the local surface fluctuations can be investigated by computing either the height-height correlation function, $G(l, t) = \langle \overline{[h(\mathbf{x} + \mathbf{l}, t) - h(\mathbf{x}, t)]^2} \rangle$, where the average is calculated over all $\mathbf{x}$ (overline) and noise (brackets), or the *local* width, $w(l, t) = \langle \langle [h(\mathbf{x}, t) - \langle h \rangle_l]^2 \rangle_l \rangle^{1/2}$, where $\langle \cdots \rangle_l$ denotes an average over $\mathbf{x}$ in windows of size l. For growth processes in which an anomalous roughening takes place these functions scale as

$$w(l, t) \sim \sqrt{G(l, t)} = t^{\beta} f_A(l/t^{1/z}), \qquad (4)$$

with an anomalous scaling function [7–9] given by

$$f_A(u) \sim \begin{cases} u^{\alpha_{\mathrm{loc}}} & \text{if } u \ll 1, \\ \text{const} & \text{if } u \gg 1, \end{cases} \qquad (5)$$

instead of Eq. (2). The standard self-affine Family-Vicsek scaling [3] is then recovered when $\alpha = \alpha_{\mathrm{loc}}$.

It is important to note that anomalous scaling stems from the fact that the mean square local slope has a nontrivial dynamics. A standard (Family-Vicsek) self-affine scaling of the local interface fluctuations, i.e., $\alpha = \alpha_{\mathrm{loc}}$, implies that the square local slope $G(l = a, t) = \langle \overline{[h(\mathbf{x} + \mathbf{a}, t) - h(\mathbf{x}, t)]^2} \rangle$, where a is the lattice spacing, becomes $G(a, t) \sim \text{const}$ very rapidly in time. One can also see that this constant must go to zero as $a \to 0$.

However, as can be easily seen from Eqs. (4) and (5), in growth models exhibiting anomalous scaling the local slopes scale as

$$G(a,t) \sim t^{2\kappa}, \tag{6}$$

where the exponent $\kappa = \beta - \alpha_{\text{loc}}/z > 0$. The existence of such a diverging mean local slope introduces a new correlation length in the growth direction, which enters the scaling of the local fluctuations of the height. Therefore, in any growth model, the existence (or absence) of anomalous scaling of the local height fluctuations can be investigated by computing the mean local slope $G(a,t)$. In the continuous limit, $a \to 0$, $G(a,t)$ can be written as $G(a,t) \simeq s(t)a^2$, where

$$s(t) = \langle \overline{(\nabla h)^2} \rangle. \tag{7}$$

This corresponds to calculating the mean square local derivative of the interface height. In general, this quantity scales as a power law $s(t) \sim t^{2\kappa}$. Negative values of the exponent κ will result in a normal Family-Vicsek scaling of the local fluctuations with the same roughness exponent. On the contrary, for $\kappa > 0$ the correlation length $s(t)$ diverges in time and becomes a relevant length scale in the problem that changes the local scaling, as discussed above. In this case, anomalous scaling with a local roughness exponent

$$\alpha_{\text{loc}} = \alpha - z\kappa \tag{8}$$

is expected to occur. As we will see in the following, this simple observation allows us to find a general method to compute anomalous critical exponents. Note also that the exponent κ corresponds to the anomalous time exponent β_* in Refs. [6,8,9].

Now, let us see how the scaling behavior of the mean local slope $s(t)$ can be obtained from the continuum growth equation. By applying the operator ∇ to both sides of the growth equation (3) one gets to an equation of motion for the local derivative $\Upsilon(\mathbf{x},t) = \nabla h$ of the form

$$\frac{\partial \Upsilon}{\partial t} = \frac{\delta \Phi}{\delta \Upsilon} \nabla \Upsilon + \nabla \eta. \tag{9}$$

In general, this Langevin equation may contain nonlinear terms that break the translational symmetry $h \to h + c$ in the growth direction. This implies that the resulting interface $\Upsilon(\mathbf{x},t)$ may not be rough ($\alpha = 0$).

The global width of the interface $\Upsilon(\mathbf{x},t)$ is given by

$$W_\Upsilon(t) = \langle \overline{\Upsilon(\mathbf{x},t)^2} \rangle^{1/2} = \langle \overline{(\nabla h)^2} \rangle^{1/2}, \tag{10}$$

where $\langle \overline{\nabla h} \rangle = 0$ has been used. This leads to the general result that $s(t) = W_\Upsilon^2(t)$ and the exponent κ of the average local interface slope $s(t) \sim t^{2\kappa}$ is given by the time exponent of the global width of the local derivative $\Upsilon(\mathbf{x},t)$,

$$W_\Upsilon(t) \sim t^\kappa. \tag{11}$$

It then becomes clear that one could obtain the anomalous exponents of the interface $h(\mathbf{x},t)$ by finding the time scaling behavior of the fluctuations of $\Upsilon = \nabla h$.

In the following we investigate the existence of anomalous scaling in several continuum growth models. In the examples that we analyze here, a Flory-type approach introduced by Hentschel and Family [14] suffices to obtain the exponent κ from the corresponding Eq. (9) for every model in good agreement with existing simulation results. The Flory approach can be seen as a stability analysis of the equation of motion (9) for the corresponding local derivative $\Upsilon(\mathbf{x},t)$ of the interface.

Kardar-Parisi-Zhang equation.—The first system we examine is the noise-driven interface growth model given by the Kardar-Parisi-Zhang (KPZ) equation [15]

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \lambda (\nabla h)^2 + \eta(\mathbf{x},t). \tag{12}$$

This equation was originally introduced to describe ballistic deposition growth far from equilibrium. In $1 + 1$ dimensions the critical exponents $\alpha = 1/2$ and $z = 3/2$ can be calculated exactly [15]. On the basis of much numerical work, it is well established that the KPZ equation does not exhibit anomalous roughening. Let us see how this result can be derived in dimension $1 + 1$ by use of our approach.

In this case, the growth equation for the local derivative, Eq. (9), reads

$$\frac{\partial \Upsilon}{\partial t} = \nu \frac{\partial^2 \Upsilon}{\partial x^2} + \lambda \frac{\partial}{\partial x} \Upsilon^2 + \frac{\partial \eta}{\partial x}, \tag{13}$$

in $1 + 1$ dimensions. From this growth equation the scaling behavior of the fluctuations of the interface $\Upsilon(x,t)$ can be obtained. We find that the width scales as $W_\Upsilon(t) \sim t^{-1/5}$, where the exponent can be obtained by a Flory-type approach [14] as follows.

In the spirit of Ref. [14], we assume that at long times $t \gg t_l$, and averaged over length scales l, the typical magnitude of the fluctuations in $\Upsilon(x,t)$ scale as Υ_l, and that these fluctuations last for times of the order t_l. The idea now is to estimate the magnitude of the individual terms. Basically for any equation such as Eq. (13) to show time scaling each separate term, when coarse grained over length scales l, must be of the same order of magnitude or negligible. Only under these circumstances can scaling behavior arise. The various terms in Eq. (13) may be estimated as $\langle |\partial \Upsilon/\partial t| \rangle_l \sim \Upsilon_l/t_l$, $\langle |\partial^2 \Upsilon/\partial x^2| \rangle_l \sim \Upsilon_l/l^2$, $\langle |\partial \Upsilon^2/\partial x| \rangle_l \sim \Upsilon_l^2/l$. As for the noise, one can estimate its fluctuations on length scales l and time scales t_l as $\langle |\partial \eta/\partial x| \rangle_l \sim (l^3 t_l)^{-1/2}$ for smooth surfaces, while for rough surfaces $\langle |\partial \eta/\partial x| \rangle_l \sim (\Upsilon_l l^2 t_l)^{-1/2}$ (see Ref. [14] for details). Note that the correct scaling of the noise term depends on the particular equation under study.

At sufficiently large length scales the nonlinear term in Eq. (13) will dominate the diffusion term, and, so, equating the $\langle |\partial \Upsilon/\partial t| \rangle_l$ term with the nonlinear term we obtain that a typical fluctuation scales as $\Upsilon_l \sim l/t_l$. To proceed further we now equate our estimate for the noise fluctuation $(\Upsilon_l l^2 t_l)^{-1/2}$ to the inertial term, which gives

$\Upsilon_l \sim (t_l/l^2)^{1/3}$. Thus we can estimate that a fluctuation of Υ scales in time as $\Upsilon_l \sim t_l^{-1/5}$.

In the case of the KPZ equation in $1 + 1$ dimensions a negative exponent $\kappa = -1/5$ is found indicating a standard scaling as expected. A similar Flory-type computation also shows that, in fact, the KPZ model exhibits a self-affine scaling in $d + 1$ dimensions and $\kappa = -1/(4 + d)$.

A particular case of the KPZ equation is the linear interface growth model ($\lambda = 0$), first studied by Edwards and Wilkinson [16]. For this model Eq. (13) can be solved exactly and we obtain the exponent $\kappa = -1/4$, as corresponds to a standard Family-Vicsek scaling.

Surface growth with conservation law.—The KPZ equation does not conserve the total volume of the interface. The conserved version of KPZ was studied by Sun, Guo, and Grant (SGG) [17] and is given by

$$\frac{\partial h}{\partial t} = -K\nabla^4 h + \lambda\nabla^2(\nabla h)^2 + \eta_c(\mathbf{x}, t), \qquad (14)$$

where

$$\langle \eta_c(\mathbf{x}, t)\eta_c(\mathbf{x}', t')\rangle = -2D\nabla^2\delta(\mathbf{x} - \mathbf{x}')\delta(t - t'). \quad (15)$$

Here the exponents are exact in any dimension [17], in particular, $\alpha = 1/3$ and $z = 11/3$ for $d = 1$. We have investigated the possibility of anomalous scaling in this model in $1 + 1$ dimensions. From the corresponding equation for the local derivative Eq. (9) and the noise fluctuations $\sim(l^3 t_l)^{-1/2}$, we obtain the scaling behavior $W_\Upsilon(t) \sim t^{-2/11}$ for the fluctuations of $\Upsilon(x, t)$. This result shows that the SGG equation has also a normal scaling of the local fluctuations of the height.

Linear MBE model.—As a simple example of anomalous roughening, we now consider the linear model for MBE growth [18,19] given by

$$\frac{\partial h}{\partial t} = -K\nabla^4 h + \eta(\mathbf{x}, t). \qquad (16)$$

Despite its simplicity, this equation has played an important role in the theory of MBE. The critical exponents are easily calculated in any dimension, and, in particular, one has $\alpha = 3/2$ and $z = 4$ in dimension $d = 1$. The model exhibits superrough interfaces ($\alpha > 1$). This leads to anomalous (superrough) scaling [8,9]. In fact, numerical simulations [8,9,18,19] of Eq. (16) showed that the local scaling is given by Eqs. (4) and (5) with a local roughness exponent $\alpha_{\text{loc}} \simeq 1$.

In this case the growth equation for the local derivative,

$$\frac{\partial\Upsilon}{\partial t} = -K\frac{\partial^4\Upsilon}{\partial x^4} + \frac{\partial}{\partial x}\eta(x, t), \qquad (17)$$

is linear and can be easily solved. The Flory approach now gives the exact exponent κ. We can estimate the curvature diffusion term as $\langle|\partial^4\Upsilon/\partial x^4|\rangle_l \sim \Upsilon_l/l^4$, being the estimate for the noise term $\langle|\partial\eta/\partial x|\rangle_l \sim (l^3 t_l)^{-1/2}$ in this case. Equating fluctuations of each of the two terms on the right-hand side of Eq. (17) with the inertial term, we obtain the scaling behavior $W_\Upsilon(t) \sim t^{1/8}$ for

the width of the local interface derivative. The positive value of the exponent κ means that the local slope $s(t)$ becomes a relevant correlation length in the problem and anomalous roughening is to be expected. The scaling relation (8) gives us an exact determination of the local roughness exponent $\alpha_{\text{loc}} = 1$ for this model in dimension $d = 1$. This is in good agreement with existing numerical results [8,18,19].

Random diffusion model.—So far we have been considering growth models in which the only source of randomness is in the influx of particles on the surface. Let us now consider the growth model

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left[D(x)\frac{\partial h}{\partial x}\right] + \eta(x, t), \qquad (18)$$

where $D(x) > 0$ is a quenched columnar disorder with no correlations. This random diffusion coefficient is distributed according to $P(D) \sim D^{-\mu}$ for $D < 1$ and $P(D) = 0$ for $D > 1$. This model describes a fluid interface advancing through a stratified porous medium with long range correlations in the growth direction.

The random diffusion model was originally introduced [6] to demonstrate that anomalous roughening is not due to superroughening effects, but it can also take place in models in which $\alpha < 1$. The critical exponents depend on the disorder strength μ and can be calculated exactly [6], $\alpha = 1/[2(1 - \mu)]$ and $z = (2 - \mu)/(1 - \mu)$ for $0 < \mu < 1$. This model exhibits anomalous roughening with a local roughness exponent $\alpha_{\text{loc}} = 1/2$ that is independent of the disorder [6].

The existence of anomalous roughening in this model can be rederived by use of our approach as follows. The growth equation for the local derivative

$$\frac{\partial\Upsilon}{\partial t} = \frac{\partial^2}{\partial x^2}[D(x)\Upsilon] + \frac{\partial}{\partial x}\eta(x, t) \qquad (19)$$

can be solved exactly to obtain the time scaling behavior of $W_\Upsilon(t) \sim t^\kappa$. We find that $\kappa = \mu/[2(2 - \mu)]$ for $0 < \mu < 1$, which gives us a local roughness exponent $\alpha_{\text{loc}} = \alpha - z\kappa = 1/2$ in agreement with earlier results [6].

Lai–Das Sarma–Villain equation.—The last example we study is the LDV [20,21] equation for MBE growth,

$$\frac{\partial h}{\partial t} = -K\nabla^4 h + \lambda\nabla^2(\nabla h)^2 + \eta(\mathbf{x}, t). \qquad (20)$$

This equation is expected to describe the behavior of the long-wavelength fluctuations of the interface height in several discrete models of MBE growth [5,7,11,21,22]. Note that this equation differs from the SGG equation discussed above in the nonconserved character of the noise in this case.

According to a dynamical RG analysis of Eq. (20) the global exponents $\alpha = (4 - d)/3$ and $z = (8 + d)/3$ are expected to be exact to all loops. However, numerical simulations of the LDV equation and several discrete growth models in the same universality class have shown that the model exhibits anomalous scaling with a local

roughness exponent $\alpha_{loc} \simeq 0.7$ [5,7,11] in $1 + 1$ dimensions. A perturbative dynamic RG approach to Eq. (20) showed [12] the existence of a strong-coupling infrared singularity that prevented one from obtaining the local anomalous behavior by perturbative methods.

Again, the existence of anomalous scaling in this model in $1 + 1$ dimensions can be investigated by use of the growth equation for the local derivative of the interface. In this case Eq. (9) reads

$$\frac{\partial \Upsilon}{\partial t} = -K \frac{\partial^4 \Upsilon}{\partial x^4} + \lambda \frac{\partial^3}{\partial x^3}(\Upsilon^2) + \frac{\partial}{\partial x} \eta(x, t). \quad (21)$$

A Flory approach can also be applied to this case to determine the scaling of $W_\Upsilon \sim t^\kappa$. As we did for the KPZ equation, we can estimate the terms in Eq. (21) as $\langle|\partial \Upsilon/\partial t|\rangle_l \sim \Upsilon_l/t_l$, $\langle|\partial^4 \Upsilon/\partial x^4|\rangle_l \sim \Upsilon_l/l^4$, $\langle|\partial^3 \Upsilon^2/\partial x^3|\rangle_l \sim \Upsilon_l^2/l^3$, and the noise being $\langle|\partial \eta/\partial x|\rangle_l \sim (\Upsilon_l l^2 t_l)^{-1/2}$ as in the KPZ case. Assuming that the nonlinear term dominates the curvature diffusion term at large scales, we obtain $W_\Upsilon \sim t^{1/11}$.

According to this the exponent $\kappa = 1/11$ is positive and, as a consequence, the LDV equation in $1 + 1$ dimensions is expected to display anomalous roughening. The local roughness exponent $\alpha_{loc} = 8/11 \simeq 0.73$ is given by the scaling relation (8) with $\alpha = 1$ and $z = 3$ for $d = 1$. Our estimate for α_{loc} is in excellent agreement with the numerical result $\alpha_{loc} \simeq 0.73 \pm 0.04$ [5].

In summary, we have introduced a new method to calculate the local roughness exponent in growth models exhibiting anomalous kinetic roughening. We have shown that a divergent dynamics of the mean square local derivative, $s(t) \sim t^{2\kappa}$ with $\kappa > 0$, is the cause for an anomalous roughening of the local surface fluctuations. The exponent κ can be obtained by studying the time scaling behavior of the fluctuations of the local derivative. The local roughness exponent can then be obtained by use of a scaling relation, Eq. (8). We have examined the existence of anomalous roughening in several models. For the linear models our results are exact. Nonlinear growth equations were studied by a scaling approach, which gave results in good agreement with existing simulations.

Finally, our findings indicate that a self-affine ($\alpha = \alpha_{loc}$) Family-Vicsek scaling of the interface fluctuations is the result of a fine balance among the relevant terms in the growth equation. This balance can be altered by including conservation laws or higher order nonlinear terms resulting in a divergent behavior of the local derivatives $s(t)$ and anomalous scaling.

*Electronic address: j.lopez@ic.ac.uk
  Present address: Dipartamento di Fisica and Unità INFM, Università di Roma "La Sapienza," I-00185 Roma, Italy.

[1] A.-L. Barabási and H. E. Stanley, *Fractal Concepts in Surface Growth* (Cambridge University Press, Cambridge, England, 1995).

[2] J. Krug, Adv. Phys. **46**, 139 (1997).

[3] F. Family and T. Vicsek, J. Phys. A **18**, L75 (1985).

[4] M. Schroeder *et al.,* Europhys. Lett. **24**, 563 (1993).

[5] J. Krug, Phys. Rev. Lett. **72**, 2907 (1994).

[6] J. M. López and M. A. Rodríguez, Phys. Rev. E **54**, R2189 (1996).

[7] S. Das Sarma *et al.,* Phys. Rev. E **53**, 359 (1996).

[8] J. M. López, M. A. Rodríguez, and R. Cuerno, Phys. Rev. E **56**, 3993 (1997).

[9] J. M. López, M. A. Rodríguez, and R. Cuerno, Physica (Amsterdam) **246A**, 329 (1997).

[10] M. Castro *et al.,* Phys. Rev. E **57**, R2491 (1998).

[11] C. Dasgupta, S. Das Sarma, and J. M. Kim, Phys. Rev. E **54**, R4552 (1996).

[12] J. K. Bhattacharjee, S. Das Sarma, and R. Kotlyar, Phys. Rev. E **53**, R1313 (1996).

[13] For most models of interest, in particular, those studied here, there is invariance under both translations along the growth and substrate direction as well as invariance in the election of the time origin. These symmetries rule out an explicit dependence of Φ on h, $\mathbf{x}$, and t.

[14] H. G. E. Hentschel and F. Family, Phys. Rev. Lett. **66**, 1982 (1991).

[15] M. Kardar, G. Parisi, and Y.-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).

[16] S. F. Edwards and D. R. Wilkinson, Proc. R. Soc. London A **381**, 17 (1982).

[17] T. Sun, H. Guo, and M. Grant, Phys. Rev. A **40**, 6763 (1989).

[18] J. G. Amar, P.-M. Lam, and F. Family, Phys. Rev. E **47**, 3242 (1993).

[19] S. Das Sarma, S. V. Ghaisas, and J. M. Kim, Phys. Rev. E **49**, 122 (1994).

[20] Z. W. Lai and S. Das Sarma, Phys. Rev. Lett. **66**, 2348 (1991).

[21] J. M. Kim and S. Das Sarma, Phys. Rev. Lett. **72**, 2903 (1994).

[22] Z.-F. Huang and B.-L. Gu, Phys. Rev. E **54**, 5935 (1996).