Multiple Attractor Bifurcations: A Source of Unpredictability in Piecewise Smooth Systems

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There exists a variety of physically interesting situations described by continuous maps that are nondifferentiable on some surface in phase space. Such systems exhibit novel types of bifurcations in which multiple coexisting attractors can be created simultaneously. The striking feature of these bifurcations is that in the presence of *arbitrarily small noise* they lead to fundamentally unpredictable behavior of orbits as a system parameter is varied slowly through its bifurcation value. This unpredictability gradually disappears as the speed of variation of the system parameter through the bifurcation is reduced to zero.

In the literature dealing with bifurcation theory, differentiable maps are commonly considered. On the other hand, maps that are piecewise differentiable occur in a variety of physical situations. We will consider here two dimensional maps. By a piecewise differentiable two-dimensional map we mean that the map is continuous and there is a curve (which we call the *border*) separating the phase space into two regions, such that the map is differentiable on both sides of the border, but not on it. In particular, in this Letter we shall be concerned with the case where the map's derivative changes discontinuously across the border. This circumstance leads to a rich class of bifurcation phenomena called *border-collision bifurcations* [1,2], examples of which may be found in instances of grazing impact in mechanical oscillators [3–7], in piecewise-linear electronic circuits [8,9], and in commercial power electronic devices [10,11].

In this Letter, we study the occurrence and prevalence of border-collision bifurcations in which multiple coexisting attractors are simultaneously created at the bifurcation. We call these bifurcations *multiple attractor bifurcations*.

Our principal point in this Letter is that multiple attractor bifurcations display a striking new type of extreme sensitivity to noise. In particular, assume that the noise δN is at all times limited by some small value $\epsilon > |\delta N|$. Assume that before the bifurcation, the orbit follows a particular attractor (to within the noise) and then imagine that, as the system orbit evolves, the bifurcation parameter is very slowly varied, eventually passing through its value at the bifurcation. The question is: After the bifurcation, which attractor does the orbit follow? We argue that *no matter how small the noise level ϵ may be,* this question may be fundamentally unanswerable [12].

There are many physical systems that demonstrate the above-mentioned phenomenon. As a simple representative example, we consider the electronic circuit with an analog switch [13], S, controlled by a comparator [14], O, as depicted in Fig. 1(a). Let q denote the electric charge on the capacitor C, and let i denote the current flowing through the inductor L and the resistor R. For

small values of the current, i, the voltage drop, iR, across the resistor R is smaller than the reference voltage, v_{ref}. Consequently, the comparator, O, keeps the switch, S, open (off). However, when the current increases so that $i > i_{crit} = v_{ref}/R$, the comparator output changes state, thereby closing the switch, S. Ideally, the transition of the switch from an open state to a closed state (and vice versa) should be sharp and instantaneous. In reality, however, the situation is somewhat compromised by a variety of factors including finite gain and slew rate in the comparator op-amp, and finite rise and fall times of the switch. This causes a slight *blurring* of the $i = i_{crit}$ boundary. Fortunately, the precision and speed afforded by present-day components restrict this boundary broadening to levels insignificant compared to the typical value of i in the circuit, thereby justifying the sharp boundary model. The entire system is driven by an externally applied voltage signal $v(t)$ which is a square wave of amplitude v_{amp} with a constant bias v_{bias}. The circuit model can be described by the following nondimensionalized equations:

$$dQ/dT = \begin{cases} I & (I \le 1), \\ I - (Q + Q_{bias})/\rho & (I > 1), \end{cases} \quad (1)$$

$$dI/dT = \begin{cases} -\Omega^2 Q - \Gamma I + F & (0 < T \le 1/2), \\ -\Omega^2 Q - \Gamma I - F & (1/2 < T \le 1), \end{cases} \quad (2)$$

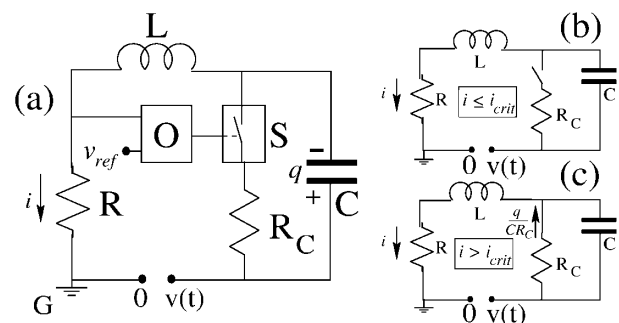where $I = i/i_{crit}$, $Q = (fq/i_{crit}) - Q_{bias}$, and the time T is normalized to the period of the square

FIG. 1. (a) The border collision circuit. (b) $i \le i_{crit}$. (c) $i > i_{crit}$.

wave drive, $T = ft$, where f is the drive frequency. The parameters in (1) and (2) are $\rho = fCR_C$, $F = v_{\text{amp}}(fLi_{\text{crit}})^{-1}$, $\Omega^{-1} = f\sqrt{LC}$, $\Gamma = R/(fL)$, and $Q_{\text{bias}} = v_{\text{bias}}fC/i_{\text{crit}}$. Figures 2(a) and 2(b) show bifurcation diagrams for this system for the two-dimensional time-1 map of this system (for parameters, see the figure caption). The interesting fact is that when (as was done to get Fig. 2) one creates bifurcation diagrams by following the attractor as the bifurcation parameter is varied, upon addition of even extremely small noise (the level of which can be judged from the small thickening of the lines in the bifurcation diagram), we find that after the bifurcation the orbit can go to different attractors: either a period-3 attractor [Fig. 2(a)] or a chaotic attractor [Fig. 2(b)].

Before proceeding further, we introduce a canonical form [15] for border-collision bifurcations given in [1], which allows us to treat piecewise-differentiable systems in a general *system-independent* manner. Let G_μ be a one-parameter family of piecewise smooth maps from the two-dimensional phase space to itself, depending smoothly on the parameter μ. Let E_μ denote a fixed point (period one orbit) of G_μ defined on $\mu_0 - \alpha < \mu < \mu_0 + \alpha$ and depending continuously on μ, for some $\alpha > 0$ (the analysis may be easily extended to include points on a period-p orbit by choosing the map to be the pth iterate of G_μ). As the parameter μ is increased from $\mu_0 - \alpha$, we suppose that the fixed point E_μ collides with the border at $\mu = \mu_0$. We could also allow the border to depend on μ, but coordinates can always be chosen so that it remains fixed. For μ near μ_0 the attractors are located near E_{μ_0}, and one can expand the map to first

order for (x, y) near E_{μ_0} and $(\mu - \mu_0)$ small. Nusse and Yorke [1] show that after suitable changes of coordinates and rescaling of the parameter, the resulting continuous, piecewise-linear map can be cast in the form

$$\hat{G}_\mu(x_n, y_n) = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{bmatrix} \tau(x) & 1 \\ -d(x) & 0 \end{bmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} \mu \\ 0 \end{pmatrix},$$

(3)

where $\tau(x)$ and $d(x)$ are piecewise constant with a discontinuity at $x = 0$, that is $[\tau(x), d(x)] = (\tau_<, d_<)$ for $x \leq 0$ and $[\tau(x), d(x)] = (\tau_>, d_>)$ for $x > 0$, where $\tau_<$, $d_<$, $\tau_>$, and $d_>$ are constants. In Eq. (3) the border has been transformed to the y axis, $\mu_0 = 0$, and $E_{\mu_0} = (0, 0)$. We refer to (3) as the *canonical form* of the original map G_μ. By the piecewise linearity of (3) in (x, y) and its linearity in μ, it is invariant under $(x, y, \mu) \rightarrow (\lambda x, \lambda y, \lambda \mu)$. Thus, as μ is reduced towards zero, any attractor of the system and its basin of attraction must contract linearly with μ, with the attractor collapsing to the point $(x, y) = (0, 0)$ [cf. Figs. 2(c) and 2(d) in which both the period-3 attractor and the chaotic attractor collapse onto the point $(0, 0)$]. Thus it appears that, no matter how small the noise level ϵ may be, there always exists a finite, positive μ interval where the attractors are close enough for the noise to induce interattractor hopping, thereby rendering prediction of the final attractor impossible [16,17].

The canonical form (3) depends on four parameters $\tau_<$, $d_<$, $\tau_>$, and $d_>$ (and the bifurcation parameter μ). To find them from the original map G_μ, we evaluate its Jacobian matrix of partial derivatives for $\mu = \mu_0$ on both sides of the border, infinitesimally close to E_{μ_0}. The parameters $\tau_<, \tau_>$ and $d_<, d_>$ are then the trace and the determinant of the Jacobian matrix on the two sides of the border. We emphasize that this approach using the canonical form may be applied to *any* generic, piecewise smooth, two-dimensional system. As an example, we evaluate the canonical form for the system, Eqs. (1) and (2), with parameters corresponding to those of Figs. 2(a) and 2(b). Figures 2(c) and 2(d) show bifurcation diagrams obtained from this canonical form (map $\hat{G}_\mu$) along with some very small additive noise. Note that this figure reproduces the particular bifurcation phenomenon: there is a border-collision bifurcation of the fixed point attractor ($\mu < 0$) to two attractors ($\mu > 0$), one a period-3 attractor and the other a chaotic attractor. This is an example of a multiple attractor bifurcation.

In our example, Eqs. (1) and (2), the Jacobian determinant is constant, $d_< = d_> = d > 0$, and for simplicity we henceforth limit our considerations to this case. Thus, for fixed d, the type of border-collision bifurcations that occurs depends on the two parameters $\tau_<$ and $\tau_>$. We explore this parameter space to assess the prevalence of multiple attractor bifurcations. Figure 3 shows the $(\tau_<, \tau_>)$ plane for $0.5 < \tau_< < 0.9$, $-2.2 < \tau_> < -1.8$, $d = 0.3$, and $\mu = 1$. Two specific initial conditions,
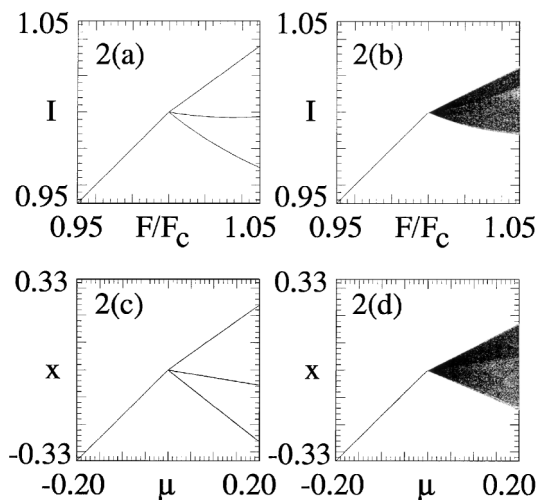

FIG. 2. (a),(b) Bifurcation diagrams for the two-dimensional time-1 map of the circuit depicted in Fig. 1 with $Q_{\text{bias}} = -1.0$, $\rho = 0.107\,42$, $\Omega = 1.0642$, and $\Gamma = 1.2040$. In creating the bifurcation diagrams (a) and (b) small noise was inserted leading to two different realizations. (c),(d) Bifurcation diagrams for the equivalent canonical form, Eq. (3), near criticality with $\tau_< = 0.7$, $\tau_> = -2.0$, and $d = 0.3$. In creating (c) and (d), small noise was inserted leading to two different realizations.
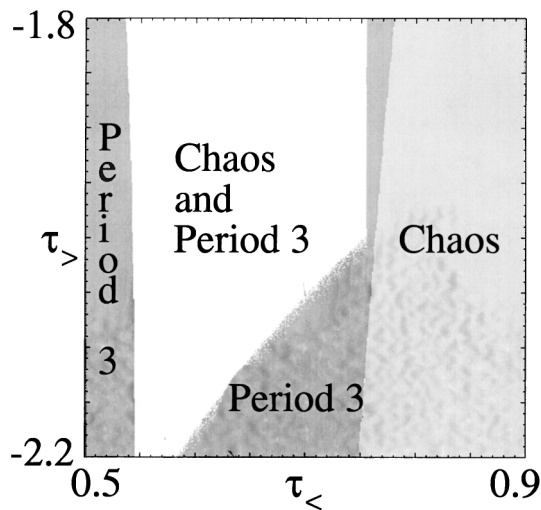
FIG. 3. Two-dimensional period plot in parameter space for the canonical form, Eq. (3), with $0.5 < \tau_< < 0.9$, $-2.2 < \tau_> < -1.8$, $d = 0.3$, $\mu = 1$, and initial conditions $(x_0, y_0) = (7.0, 2.0)$ and $(x_0, y_0) = (1.2, 0.0)$.

$(x_0, y_0) = (7.0, 2.0)$ and $(x_0, y_0) = (1.2, 0.0)$, were chosen for our simulations at every $(\tau_<, \tau_>)$ grid point. The results show four distinct regions—two where both the initial conditions lead to a period-3 attractor, another where both the initial conditions lead to a chaotic attractor, and still another where one of the initial conditions leads to a period-3 attractor while the other leads to a chaotic attractor. Clearly, the fourth region represents parameter values where at least two attractors coexist after bifurcation, thus making multiple attractor bifurcations possible.

It has been shown so far how a novel form of unpredictability might arise in piecewise smooth maps. However, this new uncertainty gradually gives way to certainty as the bifurcation speed (the rate at which the bifurcation parameter is varied in time) is reduced to arbitrarily low values. In order to understand this consider the case where there are two attractors A and B, and μ and the upper bound ϵ on the noise level are held fixed. For very small μ, the (noiseless) attractors are closer together than the noise level ϵ. For sufficiently large μ the attractors are so far apart that the noise cannot kick an orbit on one of the attractors out of its basin, and the orbit stays near the same attractor, A or B, forever. In general, however, there will be intermediate μ values μ_A and μ_B such that hopping from A to B becomes impossible when $\mu > \mu_A$, and hopping from B to A becomes impossible when $\mu > \mu_B$. Generically (in the absence of special symmetries) $\mu_A \neq \mu_B$, and we assume $\mu_B > \mu_A$; e.g., attractor B might be closer to the basin boundary than is attractor A. Therefore, if the bifurcation parameter were to be varied infinitely slowly, during the time when $\mu_B > \mu > \mu_A$, the system would certainly end up in A and it would remain there. Thus, in the quasistatic limit, there is no unpredictability as to which attractor the system ends up in.

Thus unpredictability occurs at nonzero bifurcation speed and we now ask whether we can predict anything about the relative probabilities of ending up on the different attractors given the bifurcation speed.

For simplicity, let us consider the situation with just two postbifurcation attractors, A and B, as described in a previous paragraph. Let v denote the bifurcation speed, while $a(\mu)$ and $b(\mu)$ represent the probabilities of the system being in attractors A and B, respectively. Then for small v we have

$$a + b = 1, \qquad (4)$$

$$db/dt = -\lambda_{BA} b + \lambda_{AB} a, \qquad (5)$$

where $\lambda_{AB}(\mu)$ and $\lambda_{BA}(\mu)$ are the noise induced transition probabilities from A to B and B to A, respectively. Since $\lambda_{AB}(\mu) = 0$ for $\mu > \mu_A$, we immediately have

$$b(\mu) = b(\mu_A) e^{-\frac{1}{v} \int_{\mu_A}^{\mu_B} \lambda_{BA}(\mu') \, d\mu'}, \qquad (6)$$

for $\mu > \mu_B$, where we assume that we begin at some $\mu < \mu_A$. It should be pointed out that as $v \to 0$, $b(\mu_A) \to 0$ as well, but only as v^β for $\lambda_{AB}(\mu \to \mu_A^-) \sim (\mu_A - \mu)^\beta$. Therefore, to leading order, $b(\mu > \mu_B)$, the probability of the system ending up on the nonquasistatic attractor satisfies $\ln[b(\mu > \mu_B)] \sim -1/v$.

Two sets of simulations were conducted for Eq. (3) at the parameter values corresponding to the electronic circuit described before. The system was subjected to additive noise with bounded amplitude. In one case, the noise vector distribution had the form of a uniformly filled square centered at the origin. In the other case, the distribution was a uniformly filled circular disk also centered at the origin. In both cases, when the speed of variation of the bifurcation parameter was sufficiently small, the system tended to relax to the period-3 attractor. Figure 4 confirms that, at intermediate speeds, the probability of not going to the period-3 attractor indeed conforms to Eq. (6).

In conclusion, in this paper we have discussed a novel form of uncertainty that can arise in piecewise smooth systems. We have also pointed out how that uncertainty slowly fades into certainty as the speed of bifurcation is reduced to zero.
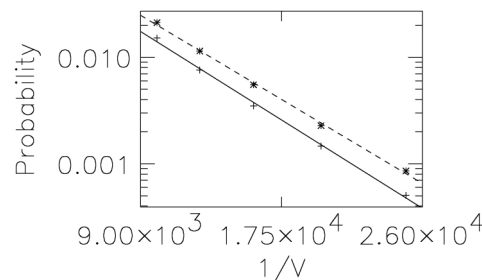

FIG. 4. Probability $b(\mu > \mu_2)$, plotted logarithmically, as a function of $1/v$, based on $1\,000\,000$ experiments for every value of $1/v$. Solid lines and plus symbols represent *square* noise, while dashed lines and asterisks represent *circular* noise.

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