Dynamical Symmetry and Higher-Order Interactions

P. Van Isacker
Grand Accélérateur National d'Ions Lourds, BP 5027, F-14076 Caen Cedex 5, France
(Received 25 June 1999)

It is shown that the concept of dynamical symmetry is enriched by increasing the order of the interactions between the constituent particles of a given many-body system. The idea is illustrated with an analysis of the interacting boson model of nuclei which shows that higher-order interactions can give rise to a rotational spectrum with O(6) dynamical symmetry.

PACS numbers: 21.60.Fw, 03.65.Fd, 21.60.Ev

A powerful method for finding eigensolutions of a quantal many-body system is based on the concepts of spectrum generating algebra and dynamical symmetry. One assumes that the Hamiltonian of the system can be written in terms of the generators of a Lie algebra, the spectrum generating algebra $G_{\rm sg}$. The latter does not, in general, define an invariant symmetry in the sense that the Hamiltonian does not commute with all generators of $G_{\rm sg}$. The invariant symmetry of the Hamiltonian (assuming it possesses one) is associated with an algebra $G_{\rm inv}$ which is a subalgebra of $G_{\rm sg}$. The enumeration of all solvable Hamiltonians amounts to finding all subalgebra embeddings from $G_{\rm sg}$ to $G_{\rm inv}$ and corresponds to the algebraic reductions

$$G_{\rm sg} \supset \left\{ \begin{array}{c} G_1^{(1)} \supset G_2^{(1)} \supset \cdots \\ \vdots \\ G_1^{(n)} \supset G_2^{(n)} \supset \cdots \end{array} \right\} \supset G_{\rm inv}\,, \qquad (1)$$

where $(1),\ldots,(n)$ enumerates all possible reductions. If the Hamiltonian can be written in terms of invariant (or Casimir) operators of a *single* reduction chain (k) in (1), its eigensolutions are known analytically. In that case the algebras $G_i^{(k)}, i = 1, 2, \ldots$ define the dynamical symmetries of the Hamiltonian: its eigenstates can be characterized by the labels associated with these algebras but, unlike an invariant symmetry, no degeneracy results.

This generic procedure is now widely used in many fields of physics with well-known examples in nuclear [1], molecular [2], and hadronic [3] physics while applications in other areas are actively researched (see [4] for a recent example in polymer physics). The procedure can perhaps best be illustrated with the example of the interacting boson model (IBM) which provides a description of collective excitations of an atomic nucleus in terms of s and d bosons [1,5]. The s and d bosons can be thought of as correlated (Cooper) pairs of valence nucleons coupled to angular momentum $l = 0$ and $l = 2$. The spectrum generating algebra of the model is U(6), generated by unitary transformations among the six components $s^\dagger$ and $d^\dagger_\mu$, since the Hamiltonian as well as other operators are expressed in terms of its generators. The invariant symmetry of the problem is defined by the angular momentum algebra O(3) since any Hamiltonian describing the nucleus must be rotationally invariant. All solvable limits of this model then correspond to algebraic reductions from U(6) to O(3). As shown by Arima and Iachello [6], there are three such limits, specified by the reductions

$$\mathrm{U}(6) \supset \left\{ \begin{array}{c} \mathrm{U}(5) \supset \mathrm{O}(5) \\ \mathrm{SU}(3) \\ \mathrm{O}(6) \supset \mathrm{O}(5) \end{array} \right\} \supset \mathrm{O}(3) \supset \mathrm{O}(2)\,, \qquad (2)$$

with spectra that are frequently observed in nuclei.

In recent years the theory of dynamical symmetries has witnessed two further developments. First, it was shown that, although (1) enumerates all *completely* solvable Hamiltonians, it is possible to construct additional interactions that preserve solvability for *part* of the eigenstates. Such situations are referred to as partial dynamical symmetries [7]. For example, in the SU(3) limit of the IBM one can construct an interaction that leaves invariant the ground band, the γ band, the double-γ $K = 4$ band, etc., but mixes all other states [8]. In a second development, it was pointed out by Kusnezov [9] and, independently, by Shirokov *et al.* [10] that the embeddings (1) are specified only up to an inner automorphism. The application of such inner automorphism on a given Hamiltonian leads to one with different interaction parameters but with the same energy spectrum. Applied to a solvable Hamiltonian, the resulting one is again solvable with the same labeling but different wave functions for its eigenstates. This observation is important for a proper understanding of the chaoticity character since it reveals the existence of hidden symmetries in certain regions of the parameter space [11].

In this Letter, it is pointed out that yet a further generalization of the idea of dynamical symmetry is possible. It represents a departure from any of the classifications in (1) but does so while preserving *some* of the dynamical symmetries $G_i^{(k)}$. An algorithm for constructing interactions with those properties is given, and examples show the existence of such interactions to be correlated with their order. The discussion will be focused on the nuclear physics example (2), but the idea may have implications in other areas of physics.

Let $G_1 \supset G_2 \supset G_3$ be a set of nested algebras which may occur anywhere in the reductions (1). The analysis that follows is independent of the exact location in

0031-9007/99/83(21)/4269(4)$15.00
© 1999 The American Physical Society

(1) of these nested algebras. It may be, for instance, that G_1 coincides with G_{sg}, or that G_3 coincides with G_{inv}, or that the entire set occurs somewhere in between G_{sg} and G_{inv}. Unless another algebra $G_2' \neq G_2$ exists such that $G_1 \supset G_2' \supset G_3$, it would appear that any Hamiltonian which has G_1 and G_3 as (dynamical) symmetries necessarily also has G_2 as a dynamical symmetry. This, in fact, is not the case, as can be shown as follows. Let $\{\hat{g}_i\}$ be the set of generators that belong to G_1 but not to G_2. Any combination of $\hat{g}_i$ cannot admix representations of G_1 (since they are generators of G_1) and hence a Hamiltonian written in terms of $\hat{g}_i$ has the labels Γ_1 associated with G_1 as good quantum numbers. It suffices now to construct combinations of $\hat{g}_i$ that are scalar in G_3 [12] (and hence conserve the associated labels Γ_3) but admix representations Γ_2 of G_2. The resulting Hamiltonian will then have the property of mixing the Γ_2 labels while having Γ_1 and Γ_3 as good quantum numbers. It does not correspond to any of the limits in (1) yet it displays the (dynamical) symmetries G_1 (and those containing G_1) and G_3 (and those contained in G_3). Note that the resulting Hamiltonian is in general not analytically solvable. Nevertheless, the occurrence of the quantum numbers Γ_1 and Γ_3 carries with it the existence of selection rules, hallmark of dynamical symmetries.

The procedure for constructing Hamiltonians with the above properties involves the identification of the tensor character under G_2 and G_3 of the operators $\hat{g}_i$, and an analysis of tensor multiplication in G_2 and G_3 and of $G_2 \supset G_3$ reduction rules. While these are evidently case dependent, they result from straightforward application of group theory.

These general notions can be exemplified by considering all nested algebras $G_1 \supset G_2 \supset G_3$ of the reductions (2). Take first the sequence O(6) $\supset$ O(5) $\supset$ O(3) which occurs in the O(6) limit of the IBM. Eigenstates in this limit are $|[N]\sigma\tau L M_L\rangle$ where the labels are associated with U(6), O(6), O(5), O(3), and O(2), respectively [1]. It is usually assumed that the goodness of the quantum number σ implies that of τ. The above analysis shows this not to be the case and, moreover, provides a procedure for generating σ-conserving, τ-violating interactions. The generators $\hat{g}_i$ contained in O(6) but not in O(5), are the five components of the quadrupole operator $\hat{Q}_\mu \equiv (s^\dagger \times \tilde{d} + d^\dagger \times \tilde{s})^{(2)}_\mu$. Combinations of $\hat{g}_i$ in this case are multiplications of $\hat{Q}_\mu$. The tensor character of $\hat{Q}_\mu$ under O(5) is $(\tau_1, \tau_2) = (1,0)$ and under O(3) evidently $L = 2$. The quadratic interaction $(\hat{Q} \times \hat{Q})^{(0)}$ corresponds to the O(5) multiplication $(1,0) \times (1,0) = (2,0) + (1,1) + (0,0)$. Since the O(5) representations (2,0), (1,1), and (0,0) contain the angular momenta $L = 2, 4$, $L = 1, 3$, and $L = 0$, respectively, it follows that the O(3)-scalar $(\hat{Q} \times \hat{Q})^{(0)}$ must also be scalar in O(5). Thus quadratic O(3)-scalar interactions that conserve O(6) necessarily also conserve O(5). This is usually taken for granted in the IBM but is seen here to be related to the order of the interaction. In the next, cubic order, the interaction $(\hat{Q} \times \hat{Q} \times \hat{Q})^{(0)}$ corresponds to $(1,0) \times (1,0) \times (1,0)$; O(5) multiplication and O(5) $\supset$ O(3) reduction rules show that there is only one O(3) scalar and it has O(5) character (3,0). Consequently, $(\hat{Q} \times \hat{Q} \times \hat{Q})^{(0)}$ is an example of a σ-conserving, τ-violating interaction; it mixes $(\tau, 0)$ with $(\tau \pm 1, 0)$ and $(\tau \pm 3, 0)$.

The analysis of $\hat{Q}$ combinations can be extended to higher orders. For example, it can be shown that the quartic interactions $(\hat{Q} \times \hat{Q} \times \hat{Q} \times \hat{Q})^{(0)}$ (there are several of them depending on the intermediate coupling) have (0,0) or (2,2) tensor character in O(5). Neither tensor can mix symmetric O(5) representations and hence such fourth-order interactions do not break O(5) dynamical symmetry.

The spectra generated by the $(\hat{Q} \times \hat{Q})^{(0)}$ and $(\hat{Q} \times \hat{Q} \times \hat{Q})^{(0)}$ interactions are different as is illustrated in Fig. 1. The top part shows the spectrum of $(\hat{Q} \times \hat{Q})^{(0)}$; it corresponds to that of a γ-soft nucleus (i.e., soft with respect to triaxial deformation) and coincides with the usual O(6) limit of the IBM. The lower part is generated by $(\hat{Q} \times \hat{Q} \times \hat{Q})^{(0)}$; the spectrum is one of γ vibrations (i.e., quadrupole vibrations around a spheroidal equilibrium

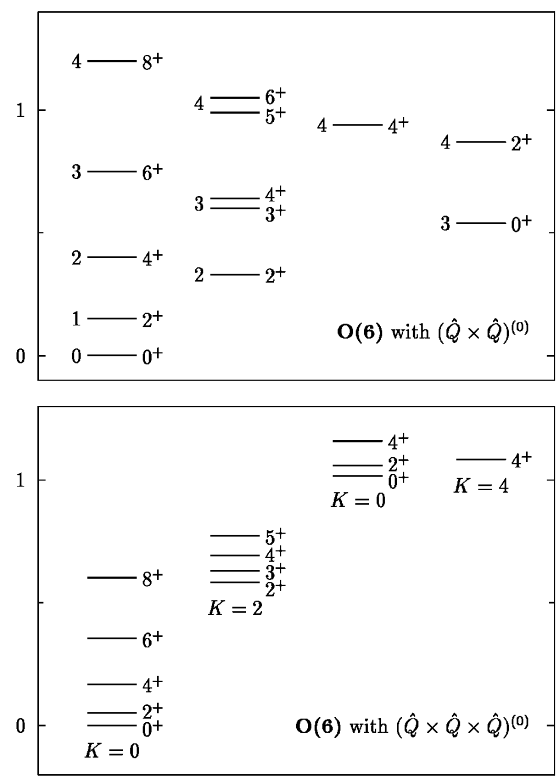

FIG. 1. Partial spectra generated by the quadratic and cubic O(6)-conserving interactions $(\hat{Q} \times \hat{Q})^{(0)}$ and $(\hat{Q} \times \hat{Q} \times \hat{Q})^{(0)}$. All levels have O(6) dynamical symmetry and those shown have $\sigma = N$. Levels are labeled by their angular momentum and parity L^π on the right. In the quadratic case states are further characterized by the O(5) label τ, shown at the left of each level while in the cubic case the K quantum number is given underneath each rotational band. Boson numbers are $N = 6$ and $N = 15$, respectively, and an $\hat{L}^2$ term is added to lift the degeneracy in L.

shape that break axial symmetry [13]) with rotational bands built on each vibration. Spectra are repeated for O(6) representations with $\sigma = N, N-2, N-4, \ldots$, but only $\sigma = N$ is shown in the figure. Eigenstates of $(\hat{Q} \times \hat{Q} \times \hat{Q})^{(0)}$ are, besides $[N]$, σ, L, and M_L, characterized by a label K (not associated with any algebra) which is interpreted as the projection of the angular momentum on the axis of symmetry. Although a formal definition of this quantum number is difficult, as it involves the solution of a third-order differential equation [14], it can be inferred from the geometric properties of *interband* and *intraband* $E2$ transitions.

There are some notable differences between the spectrum generated by $(\hat{Q} \times \hat{Q} \times \hat{Q})^{(0)}$ and that in the SU(3) limit. Specifically, in SU(3) the β ($K = 0$) and γ ($K = 2$) bands belong to the same SU(3) representation, are degenerate in energy, and are connected by strong $E2$ transitions. In the $(\hat{Q} \times \hat{Q} \times \hat{Q})^{(0)}$ spectrum, in contrast, the $K = 0$ "β" band belongs to the O(6) representation $\sigma = N - 2$ and has no $E2$ transitions to the γ band ($\sigma = N$) in the exact limit. These differences propagate to higher-lying excitations: for example, the lowest $K = 0$ two-phonon state in the $(\hat{Q} \times \hat{Q} \times \hat{Q})^{(0)}$ spectrum is a γ^2 excitation while the corresponding state in the SU(3) limit is a mixture of β^2 and γ^2 [15].

Whether cubic interactions between the bosons of IBM are required is not clear at the moment since comprehensive phenomenological studies of this question are lacking. Nevertheless, the spectrum generated by $(\hat{Q} \times \hat{Q} \times \hat{Q})^{(0)}$ has many of the properties that are typical of deformed nuclei. This is illustrated in Fig. 2 where part of the eigenspectrum obtained from a numerical diagonalization of the Hamiltonian

$$\hat{H} = \kappa(\hat{Q} \times \hat{Q} \times \hat{Q})^{(0)} + \kappa' \hat{L} \cdot \hat{L}, \qquad (3)$$

is shown for $\kappa = 2.9$ keV, $\kappa' = 8$ keV, and $N = 15$ bosons, and is compared with observed levels in ${}^{162}_{66}Dy_{96}$. The levels shown are members of the ground, γ, and the $K = 0$ and $K = 4$ bands which presumably have double-γ character; the collective nature of the latter awaits experimental confirmation from lifetime measurements [16]. Electric quadrupole transitions can be calculated with an $E2$ operator proportional to $\hat{Q}$, $\hat{T}(E2) \equiv e_b \hat{Q}$, and are shown in Table I for the γ-to-ground $E2$ transitions in ${}^{162}Dy$. A detailed, quantitative comparison with the data is not the purpose here; rather, the aim is to show that the spectrum generated by $(\hat{Q} \times \hat{Q} \times \hat{Q})^{(0)}$ has all the features of that of a deformed nucleus.

To continue the analysis of (2), the next case is the sequence of nested algebras $U(5) \supset O(5) \supset O(3)$. The set of generators $\{\hat{g}_i\}$ contained in U(5) but not in O(5) is in this case $\{\hat{T}_{00}, \hat{T}_{2\mu}, \hat{T}_{4\mu}\}$ where $\hat{T}_{k\mu} \equiv (d^\dagger \times \tilde{d})^{(k)}_\mu$. The operator $\hat{T}_{00}$ is scalar in O(5) and hence can never mix O(5) representations. The operators $\hat{T}_{2\mu}$ and $\hat{T}_{4\mu}$, on the other hand, have O(5) tensor character (2,0) and appropriate combinations of them might lead to O(5) mixing while

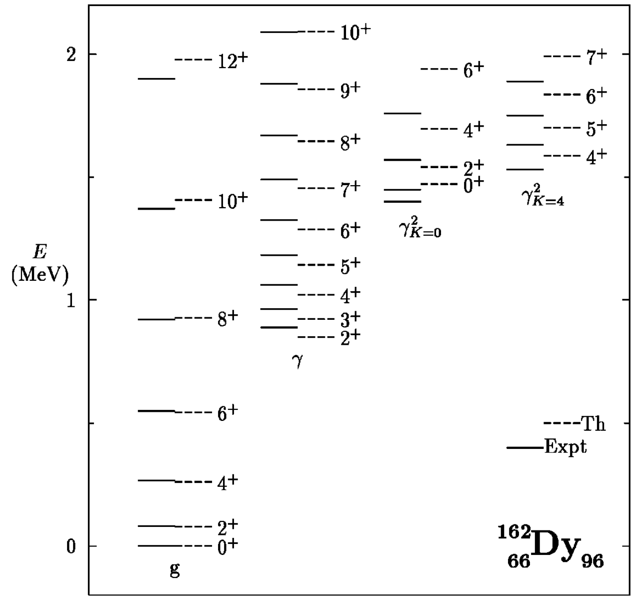

FIG. 2. Partial eigenspectrum of the Hamiltonian (3) and its comparison with the observed levels of ${}^{162}Dy$. Levels are labeled by their angular momentum and parity L^π, and all belong to the O(6) representation $\sigma = N$.

preserving the U(5) dynamical symmetry. Quadratic combinations correspond to $(2,0) \times (2,0)$ and the only O(3)-scalar terms contained in this multiplication have (0,0) or (2,2). Neither tensor induces O(5) mixing and one recovers the customary result of the IBM that for rotationally invariant two-body interactions a U(5) dynamical symmetry implies O(5). Note, however, that this result is valid only for *symmetric* O(5) representations: a (2,2) tensor can couple a symmetric representation $(\tau, 0)$ with itself but not

TABLE I. Calculated and observed γ-to-ground $E2$ transitions in ${}^{162}Dy$.

		$B(E2; L_\gamma \to L_g)$ (e^2b^2)	
L_γ	L_g	Experiment[a]	Theory
2^+_γ	0^+_g	0.024(1)	0.024[b]
	2^+_g	0.042(2)	0.039
	4^+_g	0.0030(2)	0.0026
3^+_γ	2^+_g	⋯	0.042
	4^+_g	⋯	0.023
4^+_γ	2^+_g	0.0091(5)	0.012
	4^+_g	0.044(3)	0.046
	6^+_g	0.0063(4)	0.0065
5^+_γ	4^+_g	0.033(2)	0.034
	6^+_g	0.040(2)	0.031
6^+_γ	4^+_g	0.0063(4)	0.0079
	6^+_g	0.050(4)	0.046

[a]From [17].
[b]Normalized with $e_b = 0.121$ e b.

with any other symmetric O(5) representation. This no longer is the case for nonsymmetric representations [e.g., (2,2) couples $(\tau, 0)$ with $(\tau', 1)$ and $(\tau', 2)$]. As a result, in two-component boson models such as the IBM-2 [18,19] U(5)-conserving quadratic interactions may induce O(5) mixing, a conclusion which is also found in [20] via other arguments. Cubic O(3)-scalar combinations of $\hat{T}_{2\mu}$ and $\hat{T}_{4\mu}$ can be (6,0), (4,2), (2,2), or (0,0) tensors in O(5). Of those, the first couples $(\tau, 0)$ with $(\tau \pm k, 0), k = 2, 4, 6$, while the second mixes $(\tau, 0)$ with $(\tau \pm 2, 0)$. So, in this case the conclusion is that *two* cubic d-boson interactions can be defined that induce a different O(5) mixing and hence have different spectrum generating properties.

A third sequence of nested algebras is $\mathrm{U}(6) \supset \mathrm{SU}(3) \supset \mathrm{O}(3)$. Breaking the SU(3) dynamical symmetry only leaves the trivial (dynamical) symmetries U(6) and O(3), and nothing new is obtained in this case.

Finally, the two sequences of nested algebras $\mathrm{U}(6) \supset \mathrm{U}(5) \supset \mathrm{O}(5)$ and $\mathrm{U}(6) \supset \mathrm{O}(6) \supset \mathrm{O}(5)$ can be analyzed in a similar fashion. They are examples of embeddings $G_1 \supset G_2 \supset G_3$ and $G_1 \supset G_2' \supset G_3$ with $G_2' \neq G_2$. *A priori* it is clear that the G_2 and G_2' dynamical symmetries can be broken by moving between them while conserving G_3. This was pointed out some time ago in the case of the IBM [21]: a combination of invariant operators of U(5) and O(6) does not correspond to a single reduction chain in (2) but retains O(5) as a dynamical symmetry. The same result is obtained with the method proposed here by simply noting that the d-boson number operator $\hat{n}_d$ is a scalar under O(5) yet has a (2,0) component under O(6). Consequently, $\hat{n}_d$ and combinations of it $(\hat{n}_d^2, \ldots)$ mix O(6) representations while preserving O(5) dynamical symmetry.

Two concluding remarks are in order to properly place these results into context. First, it is not the purpose here to extend symmetry-conserving interactions to higher order, as done, for instance, in [22] for the IBM. Such interactions leave all dynamical symmetries of a given classification in (1) unchanged, whereas the interactions introduced here conserve only part of them. This leads to an enrichment of the concept of dynamical symmetry, but the price to pay is loss of solvability. In the example of the IBM, the procedure gives rise to a classification in which all states have O(6) dynamical symmetry with a structure that drastically differs from the usual O(6) wave functions.

Secondly, what is proposed here is different from a partial dynamical symmetry as introduced by Alhassid and Leviatan [7]. In that work *all* (dynamical) symmetries are conserved for *part* of the eigenspectrum while here only *part* of the dynamical symmetries are conserved for *all* states. Both are partial but in a different sense. The order of the interactions plays an important role in these extensions; this is one of the results found here and is also true [23] in the case of the partial dynamical symmetries of Alhassid and Leviatan.

I wish to thank J. P. Elliott and J. A. Evans for discussions at an early stage of this work.

[1] F. Iachello and A. Arima, *The Interacting Boson Model* (Cambridge University Press, Cambridge, 1987).
[2] F. Iachello and R. D. Levine, *Algebraic Theory of Molecules* (Oxford University Press, Oxford, 1995).
[3] R. Bijker, F. Iachello, and A. Leviatan, Ann. Phys. (N.Y.) **236**, 69 (1994).
[4] F. Iachello and P. Truini, Ann. Phys. (N.Y.) **276**, 120 (1999).
[5] A. Frank and P. Van Isacker, *Algebraic Methods in Molecular and Nuclear Physics* (Wiley-Interscience, New York, 1994).
[6] A. Arima and F. Iachello, Ann. Phys. (N.Y.) **99**, 253 (1976); **111**, 201 (1978); **123**, 468 (1979).
[7] Y. Alhassid and A. Leviatan, J. Phys. A **25**, L1265 (1992).
[8] A. Leviatan, Phys. Rev. Lett. **77**, 818 (1996).
[9] D. Kusnezov, Phys. Rev. Lett. **79**, 537 (1997).
[10] A. M. Shirokov, N. A. Smirnova, and Yu. F. Smirnov, Phys. Lett. B **434**, 237 (1998).
[11] P. Cejnar and J. Jolie, Phys. Lett. B **420**, 241 (1998).
[12] In fact, the condition of scalarity under G_3 is sufficient but not necessary. It is possible that some particular nonscalar tensors do not induce any mixing of the Γ_3 representations occurring in the spectrum.
[13] A. Bohr and B. R. Mottelson, *Nuclear Structure II: Nuclear Deformations* (Benjamin, New York, 1975).
[14] J. P. Elliott (private communication).
[15] J. E. García–Ramos, C. E. Alonso, J. M. Arias, P. Van Isacker, and A. Vitturi, Nucl. Phys. **A637**, 529 (1998).
[16] A. Aprahamian (private communication).
[17] D. D. Warner, Z. R. Shi, W. Gelletly, H. G. Börner, F. Hoyler, K. Schreckenbach, and P. Van Isacker, in *Proceedings of the Sixth Conference on Capture Gamma-Ray Spectroscopy*, edited by K. Abrahams and P. Van Assche (Instute of Physics, Bristol, 1988), p. 562.
[18] A. Arima, T. Otsuka, F. Iachello, and I. Talmi, Phys. Lett. B **66**, 205 (1977).
[19] T. Otsuka, A. Arima, F. Iachello, and I. Talmi, Phys. Lett. B **76**, 139 (1978).
[20] I. Talmi, Phys. Lett. B **405**, 1 (1997).
[21] A. Leviatan, A. Novoselski, and I. Talmi, Phys. Lett. B **172**, 144 (1986).
[22] G. Vanden Berghe, H. E. De Meyer, and P. Van Isacker, Phys. Rev. C **32**, 1049 (1985).
[23] A. Leviatan *et al.* (unpublished).