Cosmological Expansion in the Presence of an Extra Dimension

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It was recently pointed out that global solutions of Einstein's equations for a 3-brane universe embedded in four spatial dimensions give rise to a Friedmann equation of the form $H \propto \rho$ on the brane, instead of the usual $H \propto \sqrt{\rho}$, which is inconsistent with cosmological observations. We remedy this problem by adding cosmological constants to the brane and the bulk, as in the recent scenario of Randall and Sundrum. Our observation allows for normal expansion during nucleosynthesis, but faster than normal expansion in the very early universe, which could be helpful for electroweak baryogenesis, for example.

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During the past year, much has been written about the possibility of having compactified extra dimensions with large radii [1]. In the original proposal, M_P was related to the radius b_0 of the N compact dimensions by $M_P^2 = M^2(Mb_0)^N$, where M is the new fundamental quantum gravity scale, which could in principle be as low as 1 TeV. If so, this would be a partial solution of the hierarchy problem, i.e., why the weak scale, M_W, is 17 orders of magnitude smaller than the Planck scale, M_P: it is because b_0 is, for some reason, much larger than M^{-1}. If $b_0 \gg M_P^{-1}$, as is necessary if $M \sim M_W$, the particles and fields of the standard model must be restricted to stay on a 3-dimensional slice (brane) of the full $N + 3$ spatial dimensions; otherwise particle propagation in the new dimensions would already have been seen in accelerator experiments. But even with the restriction of the brane, the idea implies many possibly observable effects at accelerators. It also poses severe challenges for cosmology. In this Letter we will address one of the cosmological problems, and comment upon an unexpected connection to the question of precisely how the hierarchy problem is solved using the extra dimensions.

Our starting point is the observation recently made by Binétruy *et al.* [2,3] that the Friedmann equation for the Hubble expansion rate of our 3D universe is modified, even at very low temperatures, by the presence of an extra dimension, y, compactified on a circle or an orbifold. Allowing for the possibility of a cosmological constant Λ_b in the full four spatial dimensions, called the bulk, the new Friedmann equation for the scale factor a of our brane is [2,4]

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{\rho_t}{6M^3}\right)^2 + \frac{\Lambda_b}{6M^3}, \qquad (1)$$

instead of the usual relation, $H = \sqrt{\rho_t/3M_P^2}$. ρ_t is the total (vacuum plus matter) energy density on the brane. This expression is derived, as will be explained below, from the 5D action

$$S = \int d^4x\, dy \sqrt{|g|}\left(\frac{1}{2}M^3\mathcal{R} - \Lambda_b + \mathcal{L}_{\text{brane}}\right), \quad (2)$$

where the action, $\mathcal{L}_{\text{brane}}$, for the matter living on the brane results in a stress-energy tensor parametrized as $T^\mu_\nu = \delta(by)\text{diag}(-\rho_t, p_t, p_t, p_t, 0)$. An interesting aspect of this result is the fact that, in order to find consistent global solutions to the Einstein equations in the $(4 + 1)$-dimensional spacetime, it is necessary to add a second brane [5], a mirror of our own, having equal and opposite energy density. This topology can be motivated from string theory. In the Hořava–Witten picture [6] of the nonperturbative regime of the $E_8 \times E_8$ string theory, the string coupling is interpreted as an eleventh compact dimension with a $\mathbb{Z}_2$ symmetry that truncates the spectrum in order to keep only sixteen supercharges in 10D, i.e., an $\mathcal{N} = 1$ supersymmetry in 4D after compactification on a Calabi-Yau manifold. There is good evidence [7] that over a wide range of energies the theory behaves like a 5D theory compactified on a $\mathbb{Z}_2$ orbifold with two 3-branes, viewed as the remnants of the 10D hypersurfaces where the E_8 gauge groups were living. The two 3-branes can also be seen as D3-branes of the type-I string theory [1].

Naively, one would expect that, at distances much bigger than the size of the fifth dimension, the effects of the extra compact dimension become small corrections to the usual 4D equations; thus when $H^{-1} \gg b_0$, one should recover the standard cosmology. However the presence of the mirror brane contradicts this logic. Let us choose the range of the compact coordinate to be $y \in [-1/2, +1/2]$. The solutions of the Einstein equations for the scale factor $a(y)$ behave [2,3] like $a_0(1 + A|y|/2)$, with $A \sim \rho_t$. Because the points $y = \pm 1/2$ are identified, the derivative is discontinuous at this point, and $a''/a = A[\delta(y) - \delta(y - 1/2)]$. The Einstein equations identify the delta functions with the energy densities of the two respective branes. In the limit as $b_0 \to 0$, the two branes overlap, and their energy densities cancel to first order in ρ_t because they are equal and opposite. Therefore only terms of order $(a'/a)^2 \sim A^2 \sim \rho_t^2$ survive, even at arbitrarily late times in cosmological history. The resulting expansion rate (1) is probably incompatible with big bang nucleosynthesis, which is extremely sensitive

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to how the Hubble rate varies with the energy density, hence temperature. Even if one tunes M so that the altered expansion rate (1) still gives the correct helium abundance, it is likely that the other elements will come out wrong, since their rates of production depend quite differently on the temperature.

From Eq. (1), one can imagine a very simple escape from this dilemma [8]. Suppose there is a cosmological constant Λ localized on our brane (and, correspondingly, $-\Lambda$ on the mirror brane, although this value will be corrected by terms of order ρ in the presence of matter on the branes), so that $\rho_t = \Lambda + \rho$, where ρ now denotes the energy density of normal matter or radiation on the brane, as opposed to vacuum energy. One can choose Λ_b to exactly cancel the Λ^2 terms in Eq. (1), and furthermore fix the value of Λ in terms of M and M_P:

$$\Lambda_b = -\frac{\Lambda^2}{6M^3}, \tag{3}$$

$$\Lambda = \pm 6\,\frac{M^6}{M_P^2}, \tag{4}$$

where $(\pm)$ refers to the two respective branes. Condition (3) ensures the cancellation of the effect of Λ_b by Λ^2 in (1), whereas (4) adjusts the overall rate of expansion to agree with the usual result. The new Friedmann equation then becomes the conventional one, plus a correction which is quadratic in the density:

$$H^2 = \pm\frac{\rho_\pm}{3M_P^2}\left(1 \pm \rho_\pm \frac{M_P^2}{12M^6}\right). \tag{5}$$

We have distinguished the values of ρ on the two branes by the subscript to emphasize that they need not—in fact, cannot—be the same. The brane with the positive solution has a rate of expansion that is consistent with all current cosmological observations as long as the normal rate has been recovered by the epoch of nucleosynthesis, which will be true if $0 < \rho_+ \lesssim 0.1\ \text{MeV}^4 \ll \Lambda$. One thus finds the constraint that

$$M \gtrsim 10\ \text{TeV}, \tag{6}$$

which is not much more severe than other accelerator and astrophysical limits that have recently been placed on the new gravity scale. The other brane must have $\rho_- \le 0$, since otherwise $H^2 < 0$, which has no solution.

The condition (3) is precisely what is needed to get a static universe in the case of vanishing ρ: the negative cosmological constant in the bulk cancels the positive Λ^2 from either brane. The solutions to the Einstein equations in this case were recently studied by Randall and Sundrum (RS) [9], but for very different reasons: they found that the weak scale hierarchy problem is naturally solved on one of the branes, even if $M \sim M_P$, and $b_0 \sim 50 M_P^{-1}$. This comes about because the metric tensor has an exponential dependence on the coordinate of the compact fifth dimension (see Fig. 1). Using the line element

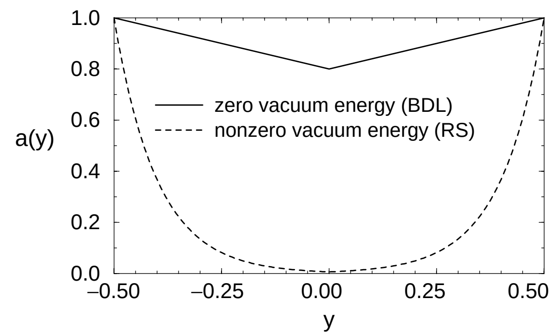

FIG. 1. Qualitative dependence of the 3D scale factor $a(y)$ on the compact dimension y in the solutions of (solid line) Ref. [2], with vanishing bulk cosmological constant, and (dashed line) Ref. [9], with Λ_b given by Eq. (3). $a(0)$ is nonzero but exponentially small in the latter.

$$ds^2 = -n^2(t,y)dt^2 + a^2(t,y)\delta_{ij}dx^i dx^j + b(t,y)^2 dy^2, \tag{7}$$

it is straightforward to verify the time-independent solution

$$a(y) = n(y) = a_0 e^{-k|y|}, \qquad k = \frac{b_0\Lambda}{6M^3}, \qquad b(y) = b_0\,. \tag{8}$$

One then observes that, even if all mass parameters in the Lagrangians formatter on the branes are of the order M_P, the physical masses on the brane at $y = 1/2$ are suppressed by the factor $e^{-k/2}$, which can be of order M_W/M_P with only a moderate hierarchy between b_0 and $M_P^{-1} \sim M^{-1}$. Since $g_{\mu\nu}$ enters differently in the kinetic than in the mass terms for a scalar field, once the kinetic terms are canonically normalized, masses get multiplied by $a(1/2) \sim e^{-k/2}$. This idea therefore appears to be a much more natural solution to the hierarchy problem than the original proposal, which required $b_0 M$ to be of order $(M_P/M)^{2/N}$, where N is the number of extra dimensions.

We now see that the static solution of RS is the starting point for our idea, which is to recover the normal expansion of the 3D universe by perturbing large, balancing cosmological constants in the bulk and the branes by a small density of matter or radiation on the branes. Intuitively, it is clear that solutions with nonvanishing ρ must exist, and we will now take some time to demonstrate this explicitly, in the vicinity of our brane. We were not able to find global solutions in closed form once matter with an arbitrary equation of state $p = \omega\rho$ was introduced. However, we are really most interested in the expansion rate on our own brane, so it suffices to solve the Einstein equations in that region. To simplify the appearance of the solutions, we will translate the y coordinate by $y \to y + 1/2$, so that the brane which we inhabit is located at $y = 0$.

We must solve the Einstein equations for the metric (7), now allowing for time dependence in a, b, and n. It is always possible to choose a gauge so that $n(t,0)$ is constant at $y = 0$, without introducing g_{05} elements in the metric.

We will make this choice, and drop all terms involving $\dot{n}$ since they are not relevant for the solution in the immediate vicinity of the brane. With this simplification, the 5D Einstein equations, $G_{\mu\nu} = M^{-3}T_{\mu\nu}$, become [2]

$$\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b}\right) = \frac{n^2}{b^2}\left[\frac{a''}{a} + \frac{a'}{a}\left(\frac{a'}{a} - \frac{b'}{b}\right)\right] + \frac{1}{3M^3}T_{00}, \tag{9}$$

$$\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{a}}{a} + 2\frac{\dot{b}}{b}\frac{\dot{a}}{a} + \frac{\ddot{b}}{b} = -\frac{n^2}{a^2M^3}T_{ii} + \frac{n^2}{b^2}\left\{2\frac{a''}{a} + \frac{n''}{n}\frac{a'}{a}\left(\frac{a'}{a} + 2\frac{n'}{n}\right) - \frac{b'}{b}\left(\frac{n'}{n} + 2\frac{a'}{a}\right)\right\}, \tag{10}$$

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{a}}{a} = \frac{n^2}{b^2}\frac{a'}{a}\left(\frac{a'}{a} + \frac{n'}{n}\right) - \frac{n^2}{3b^2M^3}T_{55}, \tag{11}$$

$$\frac{n'}{n}\frac{\dot{a}}{a} + \frac{a'}{a}\frac{\dot{b}}{b} - \frac{\dot{a}'}{a} = \frac{1}{3M^3}T_{05} = 0 \tag{12}$$

in the vicinity of $y = 0$. Close to our brane, the nonzero elements of the 5D stress-energy tensor are

$$\begin{aligned} T_{00} &= n^2(\rho + \Lambda)\delta(by) + n^2[\Lambda_b + V(b)], \\ T_{ii} &= a^2(p - \Lambda)\delta(by) - a^2[\Lambda_b + V(b)], \\ T_{55} &= -b^2[\Lambda_b + V(b) + V'(b)/b], \end{aligned} \tag{13}$$

where $\delta(by) = b^{-1}\delta(y)$ is the generally covariant form of the delta function. There are also source terms proportional to $b^{-1}\delta(y - 1/2)$ at the mirror brane, but these will not directly concern us in what follows. The terms involving $V(b)$ would result if there is a potential that stabilizes the compact dimension. Their presence does not qualitatively change any of our conclusions, so we set $V(b)$ to zero in what follows.

The generalization of the static solution (8) can be parametrized as

$$\begin{aligned} a(t,y) &= a_0(t)\exp(\tfrac{1}{2}A|y| + \tfrac{1}{2}A_2y^2 + \cdots), \\ b(t,y) &= b_0\exp(\tfrac{1}{2}B|y| + \tfrac{1}{2}B_2y^2 + \cdots), \\ n(t,y) &= \exp(\tfrac{1}{2}\mathcal{N}|y| + \tfrac{1}{2}\mathcal{N}_2y^2 + \cdots). \end{aligned} \tag{14}$$

By our choice of gauge for time, there is no $n_0(t)$ function. We have not assumed separability of the solution here, since the coefficients $A, B, \mathcal{N}$, etc., need not be static; however we will see that their time dependence arises entirely from that of ρ and p. The fact that b_0 is constant in time is not obvious, but will be proven to be consistent with Eqs. (9)–(12).

As in Ref. [2], the linear-in-$|y|$ coefficients, A and $\mathcal{N}$, are determined by the singular parts of Eqs. (9) and (10), i.e., those involving the delta functions and second spatial derivatives. One finds that

$$\begin{aligned} A &= -\tfrac{1}{3}b_0M^{-3}(\rho + \Lambda), \\ \mathcal{N} &= b_0M^{-3}(p + \tfrac{2}{3}\rho - \tfrac{1}{3}\Lambda). \end{aligned} \tag{15}$$

Therefore, to obtain solutions that are growing in the direction of the mirror brane, which are needed to solve the hierarchy problem on our own, *we would have to choose* $\Lambda < 0$ *here,* about which we shall say more below. The analogous coefficient B is not determined in this way because b'' appears nowhere in the Einstein equations. But it is constrained by Eq. (12). Inserting the ansatz (14) in this equation, and taking $\omega = p/\rho$ to be constant (which is a weak restriction since p and ρ refer only to the matter and radiation), one can eventually show that

$$B = \frac{b_0}{M^3}\left[\rho + p - \Lambda(1+\omega)\ln\left(1 + \frac{\rho}{\Lambda}\right)\right] + \mathcal{O}(A_2, \mathcal{N}_2). \tag{16}$$

and it is consistent to take $\dot{b}_0 = 0$. Thus the scale factor of the compact dimension, although it expands inside the bulk, is strictly constant on our brane. Equation (16) is not a complete specification for B since A_2 and $\mathcal{N}_2$ are not yet known, but in fact we will never need B for determining the Friedmann equation on our brane.

It remains to satisfy the nonsingular parts of the other Einstein equations, (9)–(11), near $y = 0$. The knowledge of A and $\mathcal{N}$ is all that is needed to specify Eq. (11) at $y = 0$ because no second derivatives appear. One obtains

$$\begin{aligned} \left(\frac{\dot{a}_0}{a_0}\right)^2 + \frac{\ddot{a}_0}{a_0} &= \frac{1}{36M^6}(\Lambda + \rho)(2\Lambda - \rho - 3p) + \frac{\Lambda_b}{3M^3} \\ &= \frac{\rho - 3p}{6M_P^2} - \frac{\rho(\rho + 3p)}{36M^6}, \end{aligned} \tag{17}$$

where the second equation follows from using our previous determination of Λ and Λ_b [Eqs. (3) and (4)]. The leading term reproduces the usual prediction of general relativity, and the second term corresponds to the quadratic correction in Eq. (5). Indeed, in light of the energy conservation law on the brane, $\dot{\rho} = -3H(\rho + p)$, which is true regardless of the extra dimension [2], Eq. (5) is the only relation consistent with (17) when $\omega = p/\rho$ is assumed to be constant.

In contrast to the new Einstein equation (11) associated with the fifth dimension, the G_{00} and G_{ii} equations (9) and (10) depend on the quadratic coefficients A_2 and $\mathcal{N}_2$ at $y = 0$, because of the presence of a'' and n''. With two equations in two unknowns, it is always possible to find values of A_2 and $\mathcal{N}_2$ such that the resulting equations for $a_0(t)$ are consistent with (5) and (17). Therefore Eqs. (9) and (10) add no new information on the brane, although they would be necessary if one wanted to deduce the full y dependence of the solutions in the bulk.

In the above derivation, it was shown that the brane whose masses are small by the RS mechanism must have $\Lambda < 0$. Unfortunately, we already saw in Eq. (5) that the brane with negative Λ must have an energy density $\rho_- \le 0$, which is not the case in our universe. This would appear to be a serious problem for the RS idea. However, see the "Note added" below.

The conditions (3) and (4) for the brane and bulk cosmological constants look strange at first, so some words of motivation are in order. Although when $\rho = 0$, $\Lambda_+ = -\Lambda_-$ on the two branes, when $\rho \neq 0$, a global solution to the Einstein equations is needed in order to derive the exact relation between Λ_+ and Λ_-, since it involves all of the coefficients of the expansion (14) [10]:

$$\frac{b_+(\Lambda_+ + \rho_+)}{A/2} = -\frac{b_-(\Lambda_- + \rho_-)}{A/2 + A_2 y_- + \ldots}. \qquad (18)$$

In addition to this topological relation derived from the spacetime geometry, there is also a relation involving Λ_b. We argue that the latter is a stringent consistency condition similar to the global tadpole cancellation in string theory (see, for instance, Ref. [11]):

$$\frac{\sqrt{g}}{b}\,\Lambda_{|+} + \frac{\sqrt{g}}{b}\,\Lambda_{|-} + \int_{-1/2}^{1/2} dy\,\sqrt{g}\left(\Lambda_b - \frac{1}{2}M^3\mathcal{R}\right) = 0, \qquad (19)$$

i.e., the global effective cosmological constant must vanish. In the solution of RS, this condition reduces to $\Lambda_0^2 + 6M^3\Lambda_b = 0$, which is the relation needed to obtain a global solution to Einstein equations.

The condition (19) can be understood if the cosmological constants are viewed as an effective description of the Ramond-Ramond fields of the underlying string theory: for instance the value of the $(p + 1)$ form to which a p-brane is coupled is reinterpreted as a cosmological constant on the p-brane. The condition (19) will now be necessary to cancel the UV divergences of the string theory. The connection between the phenomenological scenario of RS and string theory has recently been examined by Verlinde [12], and his analysis concludes that the exponential dependence of the metric in the compact direction is identified with the renormalization group scale when using the anti–de Sitter space/conformal field theory correspondence, which also corroborates the stringy origin of the RS mechanism.

Since the normal expansion rate of the universe is only known to have held between nucleosynthesis and the present epoch (as was stressed in Ref. [13]), it would be interesting if the quadratic corrections to the new Friedmann equation (5) started to become important above temperatures of several MeV. In the most natural version of the RS scenario, M and M_P are of the same order, so the corrections become important only at the Planck scale. However, it is still a logical possibility to imagine that the fundamental scale M is much smaller than M_P. In this case, one recovers the Arkani-Hamed *et al.* result that $M_P^2 = M^3 b_0$ which, combined with the gravitational tests that restrict $b_0 \lesssim 1$ mm, gives the constraint $M > 10^8$ GeV. With such a large value of M, departure from normal expansion occurs only above temperatures $T \gtrsim 1$ TeV, which is not far above the electroweak scale. An intriguing possibility would be to increase the rate of expansion during the electroweak phase transition. If this occurred, standard model sphaleron interactions could easily be out of equilibrium in the broken phase [13], making electroweak baryogenesis more feasible.

As we were submitting this work, Ref. [14] appeared, which reached conclusions similar to ours.

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Note added.—After acceptance of this work, we discovered [15] that the problem of the wrong-sign expansion rate at the second brane can be solved if the extra dimension is taken to be noncompact, as suggested by Ref. [16]. Reference [15] shows that by considering multiple intersecting branes, the whole construction can be extended to any number of extra dimensions.

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