Interference of Bose-Einstein Condensates in Momentum Space

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We suggest an experiment to investigate interference effects between two spatially separated Bose-Einstein condensates. In the presence of coherence, the dynamic structure factor, measurable through inelastic photon scattering, is predicted to exhibit, at high momentum transfer q, interference fringes with frequency period $\Delta\nu = q/md$, where d is the distance between the condensates. We show that this coherent configuration corresponds to an eigenstate of the physical observable measured in the experiment and that the relative phase of the condensates can be, hence, created through the measurement process.

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Dilute Bose-Einstein condensed gases behave as classical matter waves. This remarkable feature has been directly confirmed by several recent experiments [1–4]. In particular, very clean interference patterns generated by two overlapping condensates have been observed through absorption imaging techniques [1]. Interference phenomena produced by matter waves are key features underlying the quantum mechanical behavior of matter, so it is of considerable interest to understand the new role played by Bose-Einstein condensation (BEC).

Bose-Einstein condensed gases can be regarded as classical objects because, according to Bogoliubov prescription, the corresponding field operator can be replaced by a classical field, resembling the classical limit of quantum electrodynamics. Differently from the case of the electromagnetic field which is governed by the Maxwell equations, the field associated with a Bose-Einstein condensate obeys equations of quantum nature which reduce, for dilute and cold gases, to the Gross-Pitaevskii equation [5]:

$$i\hbar\frac{\partial}{\partial t}\Psi = \left(-\frac{\hbar^2\nabla^2}{2m} + V_{\rm ext} + g|\Psi|^2\right)\Psi, \qquad (1)$$

where $V_{\rm ext}$ is the external potential confining the gas, and $g = 4\pi\hbar^2 a/m$ is the interaction coupling constant, fixed by the s-wave scattering length a. The field Ψ has the meaning of an order parameter and is often called the "wave function of the condensate." As a consequence of the quantum nature of Eq. (1), key features of the field Ψ, as, for example, interference patterns, depend explicitly on the value of the Planck constant.

The fact that two overlapping condensates behave as coherent matter waves giving rise to interference is not, however, obvious. In a similar context Anderson [6] raised the intriguing question: "Do two superfluids which have never *seen* one another possess a definitive phase?" This question has been the object of theoretical speculations [7] and has been more recently reconsidered [8–12] after the experimental realization of BEC in trapped atomic gases. The point of view shared by most authors is that the relative phase between two condensates is "created" during the measurement. In other words, even if the initial configuration is not coherent and the relative phase between the condensates is not fixed, one can still observe interference in a single realization of the experiment, as explicitly proven in [1]. However, measuring fringe patterns in the density profiles requires overlapping of the condensates in coordinate space and one cannot exclude the possibility that interactions among atoms block the relative phase before measurement [10,13].

In this Letter, we propose an alternative way to investigate interference and coherence effects, by exploring the behavior of the condensate in *momentum* rather than in *coordinate* space [14]. Our proposal is stimulated by the recent experiment of [15] where, by measuring the dynamic structure factor at high momentum transfer via stimulated two-photon Bragg scattering, it was possible to investigate the momentum distribution of a single condensate. In this paper we consider two parallel trapped condensates, located at distance d along the x axis and described by the order parameters Ψ_a and Ψ_b, respectively. Geometries of this type are now becoming available via optical confinement techniques. In the first part of the work we show that the coherence between the two separated condensates can give rise to interference signals in measurable quantities such as the momentum distribution and the dynamic structure factor. In the second part we discuss the consequences of the measurement process which can bring two initially independent condensates into a coherent configuration with a well-defined relative phase.

In the presence of coherence, the order parameter of the whole system is given by the linear combination $\Psi_c = \Psi_a + e^{i\phi}\Psi_b$. The corresponding many-body function is $|c\rangle = (c^\dagger)^N|\,\rangle$, where $c^\dagger$ is the particle creation operator in the state Ψ_c, and $|\,\rangle$ is the vacuum of particles. In the following we assume, for simplicity, that the two condensates have the same shape, contain the same average number of atoms ($N_a = N_b = N/2$), and that the potential separating the two condensates is large enough to exclude overlap between the two wave functions. Under these conditions we can write $\Psi_a(\mathbf{r}) = \Psi_0(\mathbf{r} + \mathbf{d}/2)$ and $\Psi_b(\mathbf{r}) = \Psi_0(\mathbf{r} - \mathbf{d}/2)$, where Ψ_0 is the order parameter of a single condensate obeying Eq. (1) and normalized to $\int d\mathbf{r}|\Psi_0|^2 = N/2$. In momentum space, the order parameter takes the form

$$\Psi(\mathbf{p}) = e^{-ip_x d/2\hbar}\Psi_0(\mathbf{p}) + e^{i(\phi + p_x d/2\hbar)}\Psi_0(\mathbf{p}), \quad (2)$$

where $\Psi_0(\mathbf{p}) = (2\pi\hbar)^{-3/2}\int d\mathbf{r}\, e(-i\mathbf{p}\cdot\mathbf{r}/\hbar)\Psi_0(\mathbf{r})$.

Let us calculate the average value of the momentum distribution operator $\hat{n}(\mathbf{p}) = \hat{\Psi}^\dagger(\mathbf{p})\hat{\Psi}(\mathbf{p})$, where $\hat{\Psi}(\mathbf{p})$ is the field operator in momentum representation. For the coherent configuration (2) one finds the result $n(\mathbf{p}) = \langle\hat{n}(\mathbf{p})\rangle = 2[1 + \cos(p_x d/\hbar + \phi)]n_0(\mathbf{p})$ which exhibits interference fringes with period $\Delta p_x = 2\pi\hbar/d$ [16]. In this equation, $n_0(\mathbf{p}) = |\Psi_0(\mathbf{p})|^2$ is the momentum distribution of each condensate. In the absence of coherence, the many-body wave function has instead the form $(a^\dagger)^{N_a}(b^\dagger)^{N_b}|\rangle$, where $a^\dagger$ and $b^\dagger$ are the particle creation operators in the states Ψ_a and Ψ_b, respectively. This state corresponds to two independent condensates with fixed number of atoms N_a and N_b, respectively, and the average of the momentum distribution operator takes the value $n(\mathbf{p}) = 2n_0(\mathbf{p})$ which, as expected, does not exhibit interference. Measuring the momentum distribution of two separated condensates is, consequently, a natural way to determine their relative phase and to point out the occurrence of coherence.

The momentum distribution of the condensate can be investigated experimentally by measuring the dynamic structure factor at high energy and momentum transfer [17]. A useful description is provided by the impulse approximation [18]:

$$S(q,E) = \frac{m}{q}\int n(Y, p_y, p_z)\, dp_y\, dp_z\,, \quad (3)$$

which relates the dynamic structure factor to the so-called longitudinal momentum distribution $\nu(p_x) = \int n(p_x, p_y, p_z)\, dp_y\, dp_z$. In (3) E and $\mathbf{q}$ are the energy and momentum transferred by the photon to the system and $Y = \frac{m}{q}(E - q^2/2m)$ is the relevant scaling variable of the problem [19]. The vector $\mathbf{q}$ has been taken along the x direction. Impulse approximation assumes that the system, after scattering with the photon, can be described in terms of a scattered atom propagating with momentum $\mathbf{p} + \mathbf{q}$ and $(N - 1)$ atoms remaining in the unperturbed configuration. This approximation has been extensively used to analyze the momentum distribution of various classical and quantum systems, including liquids and solids, through deep inelastic neutron scattering. In particular, it has been employed to extract the condensate fraction of superfluid ^{4}He [20]. The impulse approximation is accurate at large q where one can ignore final state interaction effects which are responsible for both a shift of the peak energy with respect to the free recoil value $E_r = q^2/2m$ and for a broadening of the function $S(q,E)$. These effects are always important in the case of liquid helium where one must use sophisticated theories beyond impulse approximation in order to obtain information on the momentum distribution starting from the experimental measurement of $S(q,E)$. The situation is very different in a dilute gas where the quantum depletion of the condensate is negligible and the momentum q transferred to the sample can be large compared to the inverse of the healing length, but still small compared to the inverse of the scattering length. On this scale, final state interactions can be taken into account using Bogoliubov theory, and the impulse approximation can be shown to be accurate in realistic experimental conditions (see below).

First experimental measurements of the dynamic structure factor of a trapped Bose gas have been presented in [15] by stimulated two-photon Bragg scattering. In this experiment the condensate is illuminated by two laser beams with wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2 = \mathbf{k}_1 - \mathbf{q}/\hbar$ and frequency difference $E/\hbar$. A stimulated light-scattering process converts photon 1 into photon 2 and transfers momentum $\mathbf{q}$ and energy E to the sample. The measurement of the number of the scattered atoms then provides the value of $S(q,E)$ and, hence, for large values of q, the value of the longitudinal momentum distribution $\nu(p_x)$. In this experiment the scattered atoms were detected after switching off the trap. It is, however, possible to devise an experiment where the scattered atoms have enough energy to leave the trap so that the two condensates remain confined in their original traps after the measurement, allowing for a repetition of the experiment. The authors of [15] have also provided first theoretical estimates of final state interaction effects. The relative shift of the peak is a small effect, fixed by the ratio μ/E_r where $\mu = gn(0)$ is the chemical potential and $n(0)$ is the central density of the gas. Typical values in the experiment of [15] are $E_r \sim (20\text{–}100)\mu$, depending on the density of the sample. The broadening of $S(q,E)$ due to interactions is of the order of the chemical potential and should be compared with the broadening $\Delta E \sim \hbar q/mR_x$, fixed by the width $\hbar/R_x$ of the momentum ditribution of the condensate (Doppler broadening). Here R_x is the radius of the condensate in the x direction, determined by the Thomas-Fermi relation $\mu = m\omega_x^2 R_x^2/2$, where ω_x is the radial frequency of the harmonic potential trapping each condensate. For large values of q, the Doppler broadening, accounted for by Eq. (3), is the leading effect. This happens if [15] $E_r/\mu > 0.02(\mu/\hbar\omega_x)^2$, a condition well satisfied in the low density samples investigated in this experiment.

An important feature predicted by (2) is that, in the presence of coherence, also the dynamic structure factor exhibits interference. In fact, inserting the corresponding result for the momentum distribution into the impulse approximation expression (3), one finds

$$S(q,E) = 2[1 + \cos(Yd/\hbar + \phi)]S_0(q,E)\,, \quad (4)$$

where $S_0(q, E)$ is the dynamic form factor relative to each condensate. The fringes have a period

$$\Delta\nu = \frac{\Delta E}{2\pi\hbar} = \frac{q}{md}, \tag{5}$$

and their position is fixed by the value of the relative phase ϕ which can be consequently measured. Notice that the ratio between the Doppler width of $S(q, E)$ and the distance between two fringes scales as $\sim d/R_x$. Of course results (4) and (5) require that the experimental uncertainty in the momentum transferred by the photons be smaller than $\hbar/d$. The applicability of impulse approximation also demands that the energy of the scattered atoms, of the order of the recoil energy $\sim q^2/2m$, be larger than the height of the barrier separating the two condensates. In the following, we assume that this condition be satisfied. In this case the reflection of the scattered atom from the barrier is negligible and the dynamic structure factor $S(q, E)$ will be sensitive to the relative phase ϕ of the two condensates.

In Fig. 1 we show a typical prediction for the dynamic structure function in the absence (dashed line) and in the presence of coherence (full line) between the two condensates. The curves have been calculated in impulse approximation for a gas of sodium atoms trapped in two identical and parallel cigar shaped harmonic traps with frequencies $\nu_z = 10$ Hz, $\nu_\perp = 100$ Hz, and central density equal to 0.5×10^{14} cm^{-3}. The distance between the centers of the trap is $d = 4R_x \sim 33$ μm and the momentum transfer q is equal to $21.3\hbar(\mu\text{m})^{-1}$. With such parameters the ratio between the recoil energy E_r and the chemical potential μ is 130 and the broadening of the dynamic structure factor due to final state interactions can be shown to be negligible.

It is interesting to compare the interference fringes exhibited by the momentum distribution with the ones characterizing the density of two expanding and overlapping condensates [1]. A simple estimate of the density fringes is obtained by neglecting interaction effects between the two condensates during the expansion. This is a good approximation if the two condensates are initially well separated in space. In this case one can show (see [21] and references therein) that for large times the density profile consists of straight line fringes orthogonal to the x axis and with wave length equal to $2\pi\hbar t/md$. Such fringes have been directly observed in [1].

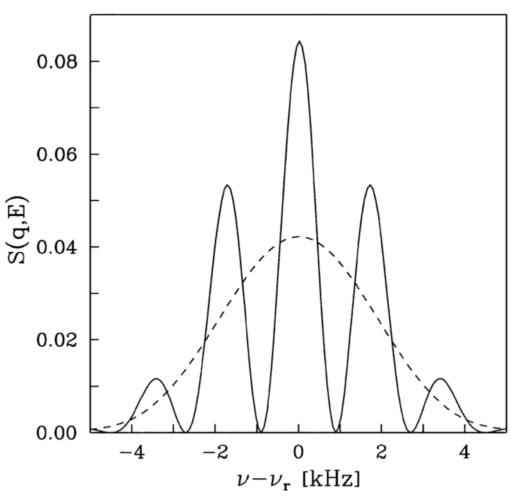

FIG. 1. Dynamic structure factor calculated with (full line) and without (dashed line) coherence between the two condensates. The relative phase was taken equal to zero, and $\nu = E/2\pi\hbar$.

The fringes of the dynamic structure factor shown in Fig. 1 are the consequence of the coherence assumed for the order parameter (2). In the remaining part of this Letter, we show that even if the two condensates are initially independent and their relative phase is not fixed, the measurement of the dynamic structure factor $S(q, E)$ will bring the system into a coherent configuration. According to quantum theory, the system, after measurement, jumps into an eigenstate of the measured observable. So the crucial point is to show that the coherent state $|c\rangle$, differently from the initial state built with two independent condensates, is an eigenstate of the longitudinal momentum operator $\nu(p_x)$, i.e., the quantity measured in the proposed experiment.

According to the Bogoliubov prescription, a fully Bose-Einstein condensed state is not only eigenstate of the field operator $\hat{\Psi}$, but also of the density as well as of the momentum density operators. Of course this assumption does not apply to configurations exhibiting fragmentation of Bose-Einstein condensation, as happens in the case of two independent condensates. Furthermore, it ignores microscopic fluctuations arising from the non-commutativity of $\hat{\Psi}$ and $\hat{\Psi}^\dagger$. Let us discuss in a quantitative way the conditions of applicability of the Bogoliubov prescription. For a proper discussion of the problem it is crucial to consider macroscopic coarse grained averages of the physical observable. This averaging takes into account the finite resolution of the experimental apparatus. On the other hand, integrating the signal can be crucial in order to produce visible interference patterns. From the theoretical side, one can show that only by taking these averages will the fluctuations of the physical observables become negligible. We will consider here the problem of the momentum distribution which is the main object of the present work. A similar discussion can be repeated for the fluctuations of the density operator in the context of experiments where two expanding condensates overlap in coordinate space and are then imaged. Since the momentum distribution enters the relevant expression (3) integrated with respect to p_y and p_z, one needs only to consider the coarse grained average along the x direction:

$$\hat{\nu}_\beta(p_x) = \frac{1}{\beta\pi^{1/2}} \int d\mathbf{p}'\, \hat{n}(\mathbf{p}') \exp[-(p_x - p_x')^2/\beta^2], \tag{6}$$

where $\hat{n}(\mathbf{p}) = \hat{\Psi}^\dagger(\mathbf{p})\hat{\Psi}(\mathbf{p})$ is the momentum distribution operator and, for simplicity, we have chosen a Gaussian convolution. The quantity (6) is expected to be a reasonable representation of the physical observable characterizing the measurement of the dynamic structure factor at high momentum transfer. The fluctuations of the operator $\hat{\nu}_\beta(p_x)$ are easily calculated for the coherent configuration. After ordering the field operators, one finds the result:

$$\langle \hat{\nu}_\beta(p_x)\hat{\nu}_\beta(p_x)\rangle - \langle \hat{\nu}_\beta(p_x)\rangle^2 = \frac{1}{\sqrt{2\pi}\,\beta}\langle \hat{\nu}_{\beta/\sqrt{2}}(p_x)\rangle,$$

where the term in the right-hand side arises from the noncommutativity of the field operators $\hat{\Psi}$ and $\hat{\Psi}^\dagger$. This equation explicitly shows that the fluctuations diverge when $\beta \to 0$. Vice versa they become negligible in the macroscopic limit $\beta\nu(p_x) \gg 1$, i.e., if the number of atoms with momentum between p_x and $p_x + \beta$ is sufficiently large. This condition is easily satisfied for actual Bose-Einstein condensates and is compatible with the request that β be smaller than the distance $\Delta p_x = 2\pi\hbar/d$ between two consecutive fringes of the momentum distribution. As a consequence of the strong quenching of the fluctuations, the coherent state $|c\rangle$ can be considered, with proper accuracy, an eigenstate of the operator $\hat{\nu}_\beta(p_x)$. The situation is very different if one instead considers two independent condensates occupying, respectively, the single particle wave functions Ψ_a and Ψ_b. In this case the randomness of the relative phase produces the additional contribution $\langle \hat{\nu}_\beta(p_x)\rangle^2/2$ to the fluctuations of $\hat{\nu}_\beta(p_x)$ which become macroscopically large.

The above discussion reveals that a state made of two independent condensates is not an eigenstate of the macroscopic observable (6). As a consequence of the measurement of ν_β, the system will jump into the coherent state $|c\rangle$ and will exhibit interference patterns. The possibility of observing interference in a single realization of the experiment is a peculiar consequence of Bose-Einstein condensation. Of course one should carry out the measurement within times shorter than the decoherence time [9,12]. Different realizations of the experiment on independent condensates would instead give rise to different values of the phase and, consequently, to strong fluctuations in the measured signal.

In conclusion, we have pointed out the occurrence of interference phenomena in momentum space exhibited by Bose-Einstein condensates separated in coordinate space. The interference patterns should be visible in the dynamic structure factor measured in photon scattering experiments at high momentum transfers.

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[1] M. R. Andrews, C. G. Townsend, H.-J. Miesner, D. S. Durfee, D. M. Kurn, and W. Kettele, Science **275**, 637 (1997).
[2] D. S. Hall, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. **81**, 1543 (1998).
[3] B. P. Anderson and M. A. Kasevich, Science **282**, 1686 (1998).
[4] L. Deng, E. W. Hagley, J. Wen, M. Trippenbach, Y. Band, P. S. Julienne, J. E. Samsarian, K. Helmerson, S. L. Rolston, and W. D. Phillips, Nature (London) **398**, 218 (1999).
[5] E. P. Gross, Nuovo Cimento **20**, 454 (1961); L. P. Pitaevskii, Sov. Phys. JETP **13**, 451 (1961).
[6] P. W. Anderson, *Basic Notions of Condensed Matter Physics* (Benjamin, Menlo Park, California, 1984).
[7] A. J. Legget and F. Sols, Found. Phys. **21**, 353 (1991).
[8] J. Javanainen and S. M. Yoo, Phys. Rev. Lett. **76**, 161 (1996).
[9] J. I. Cirac, C. W. Gardiner, M. Naraschewski, and P. Zoller, Phys. Rev. A **54**, R3714 (1996); A. Röhrl, M. Naraschewski, A. Schenzle, and H. Wallis, Phys. Rev. Lett. **78**, 4143 (1997); H. Wallis, A. Röhrl, M. Naraschewski, A. Schenzle, and H. J. Miesner, J. Mod. Opt. **44**, 1751 (1997).
[10] S. M. Burnett, K. Burnett, and J. A. Vaccarro, J. Res. Natl. Inst. Stand. Technol. **101**, 593 (1996).
[11] T. Wong, M. J. Collett, and D. F. Walls, Phys. Rev. A **54**, R3718 (1996).
[12] Y. Castin and J. Dalibard, Phys. Rev. A **55**, 4330 (1997).
[13] D. S. Rokhsar, cond-mat/9812260.
[14] Other optical methods to measure the relative phase of two condensates have been suggested by J. Ruostekoski and D. F. Walls, Phys. Rev. A **56**, 2996 (1997).
[15] J. Stenger, S. Inouye, A. P. Chikkatur, D. M. Stamper-Kurn, D. E. Pritchard, and W. Ketterle, Phys. Rev. Lett. **82**, 4569 (1999).
[16] The occurrence of interference patterns along the momentum axis of the Wigner function of two separated condensates in the independent particle model was pointed out by M. Naraschevski, A. Röhrl, H. Wallis, and A. Schenzle, Mater. Sci. Eng. B **48**, 1 (1997).
[17] Another approach to measure the momentum distribution, based on scattering of fast atoms, has been recently suggested by A. B. Kuklov and B. V. Svistunov, Phys. Rev. A **60**, R769 (1999).
[18] P. C. Hohenberg and P. M. Platzman, Phys. Rev. **152**, 198 (1966).
[19] G. B. West, Phys. Rep. **18C**, 264 (1975).
[20] P. Sokol, in *Bose-Einstein Condensation,* edited by A. Griffin, D. Snoke, and S. Stringari (Cambridge University Press, Cambridge, England, 1995), p. 51.
[21] F. Dalfovo, S. Giorgini, L. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).