The Helicoid versus the Catenoid: Geometrically Induced Bifurcations

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The minimal surfaces bounded by a frame formed of a double helix and two horizontal rods are studied. The vibration equation shows that the helicoid is the stable surface when its winding number is small. The catenoid is locally isometric to the helicoid so that their vibration spectra are strongly related. While the catenoid is known to undergo a discontinuous transition to two disks, the helicoid is shown to become unstable through a continuous transition to a ribbon-shaped surface obtained experimentally, numerically, and analytically in the limit of infinite height. The normal forms of the bifurcations confirm the analysis.

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Minimal surfaces are found in many fields of physics which range from soap films [1], lipid-water solutions [2,3], diblock copolymers [4], crystallography [5,6], protein structures [6,7], to smectic-A [8], smectic-Q [9], or blue phases [10]. These studies are often morphological and most of the time are not concerned with the stability of the surfaces, although morphological transitions can result from the lack of stability. The helicoid and the catenoid are conjugate minimal surfaces through the Bonnet transformation. They are found in protein structures [6,7]: the β-sheets may lie on a catenoid or a helicoid depending on the protein conformation. The helicoid appears in macromolecules such as DNA or in screw dislocations of smectic A [8]. The catenoid, which has been widely investigated, is the minimal surface bounded by two coaxial rings. When the distance between the rings is increased, the catenoid disappears and is replaced by two disks [11] through a discontinuous transition. Recently, its vibration spectrum has been experimentally studied with smectic films in [12] and it has been shown to be strongly related to the helicoid vibration spectrum [13]. A natural question comes about the transitions that the helicoid might undergo when varying its dimensions.

In this Letter, we study the minimal surfaces lying on a frame consisting of a double helix of vertical axis, $\mathbf{r}_\pm(s) = [\pm r\cos\phi, \pm r\sin\phi, p/(2\pi)\phi]$, $\phi \in (0, \phi_0)$, and two horizontal rods $\mathbf{r}(t) = (rt, 0, 0)$, $t \in (-1, 1)$ and $\mathbf{r}(t) = [rt\cos\phi_0, rt\sin\phi_0, p/(2\pi)\phi_0]$, $t \in (-1, 1)$ (see Fig. 1). When varying the frame, a continuous transition to a ribbon-shaped surface occurs, contrary to the catenoid case. As pointed out in [14], continuous families of minimal surfaces are useful for complete morphological studies. However, when a bounding frame is considered, it is difficult to find such families. In fact, the plateau problem (finding the minimal surfaces bounded by a given frame) is mathematically intractable in general.

Let us first derive the vibration equation of a minimal surface, and introduce some useful notation (see [15]). To define a surface without ambiguity, one needs to know two tensors called the fundamental tensors I and II:

$$a_{\alpha\beta} = \mathbf{r}_{,\alpha}\mathbf{r}_{,\beta} \quad \text{and} \quad d_{\alpha\beta} = \mathbf{r}_{,\alpha\beta}\mathbf{N}\,, \tag{1}$$

$\mathbf{N}$ being the normal of the surface at a point $\mathbf{r}$. Roughly speaking, $a_{\alpha\beta}$ gives the direction of the two tangents, while the second tensor is related to their derivatives with respect to the chosen coordinates α and β. If fluctuations of the surface occur (spontaneous ones or forced exterior ones), each point is displaced as follows:

$$\mathbf{r}'(u,v,t) = \mathbf{r}(u,v) + w(u,v,t)\mathbf{N}(u,v)\,. \tag{2}$$

We ignore the tangential displacement, which means simply a reparametrization of the surface without consequence for the balance of energy. The perturbed area of the surface element $dS = [\det(a_{\alpha\beta})]^{1/2}\,du\,dv$ is then changed into (see [16])

$$dS' = \Big(1 + Kw^2 + \frac{1}{2}D^\alpha w D_\alpha w - Kw(d^{-1})^{\alpha\beta}D_\alpha w D_\beta w - Kw^2 D^\alpha w D_\alpha w - \frac{1}{8}(D^\alpha w D_\alpha w)^2\Big)dS\,. \tag{3}$$

In our case, since the surface is minimal, the mean curvature H vanishes, so that there is no linear term in

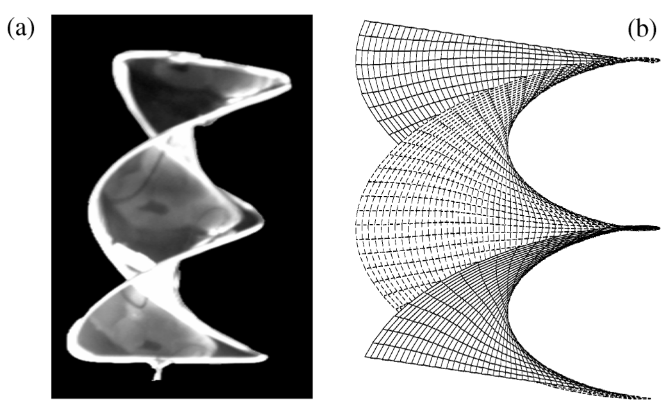

FIG. 1. Experimental and numerical helicoid for an aspect ratio $H/D = 2$ and a winding number $\phi_0/\pi = 2$.

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Eq. (3), while the Gaussian curvature K remains negative everywhere ($K = 0$ in the planar case). For the vibration equation, the second order is enough but one needs to keep the fourth order terms for the bifurcation analysis. The capillary energy E_{cap} is written as

$$E_{\text{cap}} = \gamma \int dS', \tag{4}$$

where γ is the surface tension, while the kinetic energy E_{kin} is simply

$$E_{\text{kin}} = \frac{\rho h}{2} \int (\partial_t w)^2 \, dS, \tag{5}$$

with ρ the mass density and h the thickness of the film.

With a forced harmonic excitation in $\cos \varpi t$, one gets the eigenvalue vibration equation for an arbitrary minimal surface

$$\left(D^\alpha D_\alpha - 2K + \frac{\rho h}{\gamma} \varpi^2 \right) w = 0, \tag{6}$$

with the condition that w must vanish on the frame which supports the membrane. This two-dimensional Schrödinger-like equation depends only on the metric of the surface (so on the first tensor) as the curvilinear Laplacian and the Gaussian curvature do. So if two minimal surfaces have the same metric (and differ only by the second fundamental tensor), their eigenvalue spectra are related if they rest on "similar contours." In fact, the Weierstrass construction gives an infinite set of minimal but also isometric surfaces (each of them being parametrized by a complex number $\exp i\tau$), from two analytical functions of $\omega = \theta + i\phi$. So from now on, we choose as coordinates α and β the conformal coordinates θ and ϕ. One of the most famous families is probably the heltocat family which includes as particular cases both the catenoid and the helicoid. In a previous paper [13], we have studied the isospectrality of these surfaces but now we focus on the stability analysis.

The surface is unstable if $\varpi^2 < 0$. The threshold of instability of an arbitrary minimal surface is then given by the existence of a solution to Eq. (6) with $\varpi = 0$ and the condition that w must vanish on the frame. A quick analysis of Eq. (6) with $\varpi = 0$ suggests that the helicoid threshold is simply deduced from the well-known catenoid threshold. But, of course, this provides no information on the final shape above the bifurcation threshold and on the nature of the transition, which is subcritical for the catenoid.

As a consequence, other physical effects than capillarity can intervene at the transition such as hydrodynamics which lead to a finite-time singularity [11]. Our objective here is more modest as we take into account only capillarity. Since the helicoid is a surface which occurs in a wide range of physics going from soft matter physics up to macromolecules in biology, we think it important to understand also its destabilization properties as we increase its height at constant radius. Structural transformations experienced by macromolecules might be explained simply by stability arguments.

In conformal coordinates, the representation of the heltocat family is

$$\begin{aligned} \mathbf{r}_\tau(\phi, \theta) = A(& \cos\tau \cos\phi \cosh\theta + \sin\tau \sin\phi \sinh\theta, \\ & \cos\tau \sin\phi \cosh\theta - \sin\tau \cos\phi \sinh\theta, \\ & \theta \cos\tau + \phi \sin\tau), \end{aligned} \tag{7}$$

where A is some length. By definition, all these surfaces which include as a particular case the catenoid for $\tau = 0$ and the helicoid for $\tau = \pi/2$ have the same metric and the same Gaussian curvature. Moreover, in this case, these quantities depend only on the θ variable which is, of course, a rather exceptional simplifying property for a minimal surface. Note that ϕ varies between 0 and 2π for a complete catenoid but can vary between 0 and ϕ_0 for the helicoid. As for θ, we choose $-\theta_0 < \theta < +\theta_0$. Finally, after a Fourier decomposition in ϕ, Eq. (6) reduces to the following one-dimensional equation

$$w'' + \frac{2}{\cosh^2\theta} w = \lambda^2 w. \tag{8}$$

For the catenoid, since we have periodic boundary conditions, $\lambda = 0, 1, 2, \ldots$. But for the helicoid, $\lambda = n\pi/\phi_0$ with $n = 1, 2, 3, \ldots$.

In the catenoid case with $\lambda = 0$, an analytical solution for $w = W(\theta \tanh\theta - 1)$ gives the threshold of stability $\theta_c = 1.199$ [solution of $w(\theta_c) = 0$]. So if H is the distance between the two rings of diameter D which support the catenoid, the catenoid shape disappears when the ratio H/D is larger than $(H/D)_c = 0.662$. Above this critical value, two disconnected disks are observed and the bifurcation is subcritical [11].

We now turn our attention to the helicoid surface. Equation (6) is no more than the Schrödinger equation with the attractive potential hole of modified Pöschl-Teller type [17]. The even solution, giving the lowest bound state, is found in terms of hypergeometric functions,

$$w_e(\theta) = W \cosh^2\theta F\left(\frac{2-\lambda}{2}, \frac{2+\lambda}{2}, \frac{1}{2}, -\sinh^2\theta \right). \tag{9}$$

Not surprisingly, the marginal stability is obtained for a critical θ_0 larger than 1.199. For large θ_0 an asymptotic analysis of Eq. (9) gives $\lambda = 1$. For intermediate values of θ_0, one can refer to Fig. 2 for the aspect ratio: H/D as a function of the twist angle ϕ_0/π. The domain of existence of the helicoid follows simply. Consider a helicoid of height $H = A\phi_0$ and diameter $D = 2A \sinh\theta_0$ both fixed. When increasing the twist angle ϕ_0, the helicoid becomes unstable.

Remaining at a demonstrative and qualitative level, we have observed this instability experimentally. Using iron threads we constructed a frame with two symmetrical helices with the same vertical axis and closed at both ends by horizontal rods. We used a soap solution to generate a film standing on the frame. When the twist angle is increased, an instability occurs with the destruction of

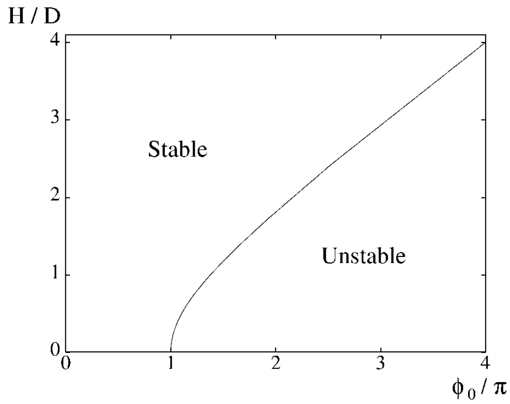

FIG. 2. Phase diagram of the helicoid.

the helicoid and the appearance of a new surface (of ribbon type) which lies on the helices (see Fig. 3). Each time, there are two possible symmetric ribbons. If we consider the DNA bases to be on a surface, the *A*-DNA [18] and the *P*-DNA [19] are morphologically similar to the ribbon surface. This simple model based on minimal surfaces roughly predicts the geometrical aspect ratios of DNA molecules and their topological changes [20]. When doing the experiment, we observe that the process is reversible: the bifurcation is supercritical. Decreasing the twist angle, the initial helicoid is restored.

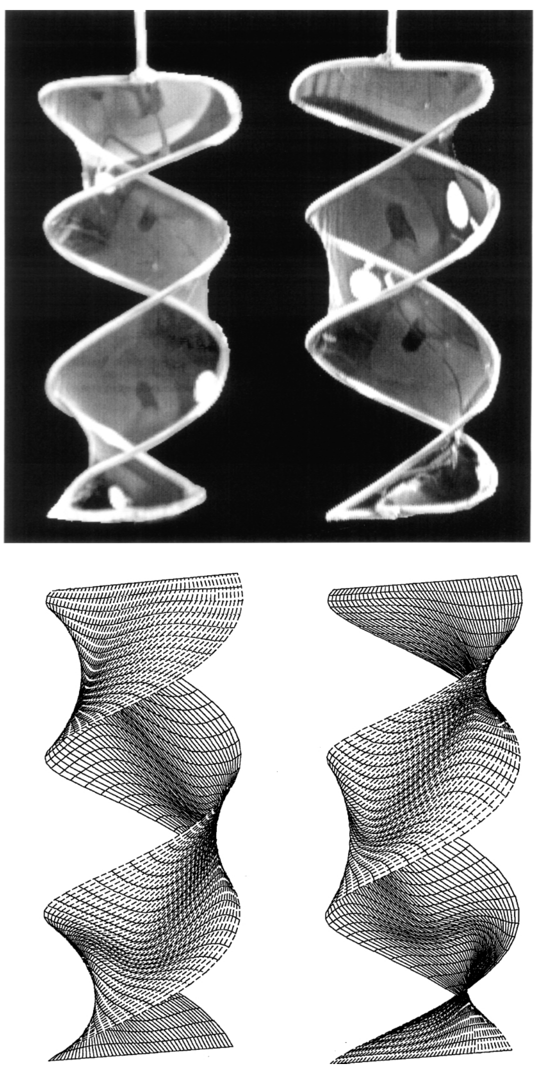

FIG. 3. The two ribbon surfaces for an aspect ratio $H/D = 2$ and a winding number $\phi_0/\pi = 3$.

We do not succeed to find explicitly the ribbon shape (bifurcated shape) by the Weierstrass construction, because of the difficulty coming from the boundary conditions at the horizontal rods. So, we use a finite-element minimization [21] of the capillary energy given the boundary conditions (the two helices and the two horizontal rods). In the domain predicted by the linear stability, the helicoid is found as expected. Outside this domain, the two ribbon surfaces are the stable minimal surfaces (Fig. 3). Note that the vertical axis lies within the helicoid while it remains outside the bifurcated surfaces. So the inner diameter d can be chosen as the order parameter for the bifurcation ($d = 0$ for the helicoid), while the control parameter can be θ, ϕ or some aspect ratio. Near the marginal stability curve, and as expected for a supercritical bifurcation, d follows the scaling $d \sim \sqrt{\lambda - \lambda_c}$ if λ is the control parameter.

This analysis can be confirmed by including the nonlinear contributions derived from Eq. (3). For clarity, we consider only the catenoid or the helicoid with $\phi_0 = \infty$, so that the most unstable mode depends on one variable θ. We compute the capillary and the kinetic energy of $w(\theta, t) = W(t)(\theta \tanh\theta - 1)$ in the vicinity of the onset of instability:

$$E_{\text{cap}} = \gamma\left(a(\theta_c - \theta_0)W^2 - g\,\frac{b}{A^2}\,W^3 + \frac{c}{A^2}\,W^4\right), \quad (10)$$

$$E_{\text{kin}} = \rho h d A^2(\partial_t W)^2. \quad (11)$$

Here a, b, c, d are positive constants. The coefficient of the second tensor $g = d_{\phi\phi} = A\cos(\tau)$ is different for the catenoid ($g = A$) and the helicoid ($g = 0$). For the catenoid, the control parameter is given by the height over diameter ratio $\delta = (H/D)_c - H/D \sim (\theta_0 - \theta_c)^2$. After a proper rescaling of the time t and amplitude W, we derive the following amplitude equation for the catenoid: $\partial_{tt}W = 2\sqrt{\delta}\,W - W^2$. By the change of function, $W = \sqrt{\delta} + Z$, the amplitude equation becomes

$$\partial_{tt}Z = \delta - Z^2, \quad (12)$$

which demonstrates that the catenoid undergoes a Hamiltonian saddle-node bifurcation. Remember that if $H/D < (H/D)_c = 0.662$, there are two possible catenoids bounded by two coaxial rings of diameter D and distant by H. One is stable and the other is unstable. The control parameter for the infinite height helicoid is given by the diameter over pitch ratio $\epsilon = D/2\pi A - (D/2\pi A)_c = (\sinh\theta_0 - \sinh\theta_c)/\pi \sim \theta_0 - \theta_c$. With the rescaling, the equation for the helicoid becomes

$$\partial_{tt}W = \epsilon W - W^3. \quad (13)$$

So the helicoid undergoes a pitchfork bifurcation. While the first tensor is responsible for the stability threshold, the second tensor is responsible for the nature of the bifurcation. In fact, the complete helicoid separates the space into two parts symmetrical with respect to the helicoid axis, so that we expect a symmetrical bifurcation. For the catenoid the two parts of space are different.

If the height of the double helix is infinite, it turns out that a simple analytical form of this new ribbon-like minimal surface can be found among the heltocat family itself. Let us consider an arbitrary member of this family Σ_τ given by Eq. (7) with B replacing A. At fixed θ, $\mathbf{r}_\tau(\phi)$ is a helix which lies on Σ_τ. Its radius is $B\sqrt{\cos^2\tau\cosh^2\theta + \sin^2\tau\sinh^2\theta}$ and its pitch is $p = 2\pi\sin\tau B$. We want to find on Σ_τ two helices of radius $r = A\sinh\theta_0$, of pitch $2\pi A$, and symmetrical with respect to the z axis, so that the frame is the helicoid frame. The second condition gives $A = B\sin\tau$, while the first and third conditions give the τ value once θ_0 is known:

$$\tan\left(\frac{\nu}{\tan\tau}\right)\tan\tau\tanh\nu = 1\,, \qquad (14)$$

$$\text{with } \sinh^2\nu = \sin^2\tau\sinh^2\theta_0 - \cos^2\tau\,. \qquad (15)$$

Note that $\theta_0 \sim 1.199 = \theta_c$ when $\tau \sim \pi/2$, and so the surface appears continuously when the helicoid becomes unstable. Also, $\theta_0 \to \infty$ when $\tau \to 0$. As given by Eq. (8), this surface is found to be always stable. But there may be many minimal surfaces bounded by a given frame. Comparison with numerical results is necessary, and shows that the surface we have found is the selected one: the numerical and theoretical surfaces differ only in the region where the vertical distance to the horizontal rods is smaller than the pitch. This difference comes from boundary effects at the rods.

In conclusion, we have shown that the helicoid undergoes a supercritical transition to a ribbon-shaped surface. This minimal surface might be used to construct a new type of dislocations, as screw dislocations are constructed from helicoids [8].

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