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Finite Precision Measurement Nullifies the Kochen-Specker Theorem

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Only finite precision measurements are experimentally reasonable, and they cannot distinguish a dense subset from its closure. We show that the rational vectors, which are dense in S^2, can be colored so that the contradiction with hidden variable theories provided by Kochen-Specker constructions does not obtain. Thus, in contrast to violation of the Bell inequalities, no quantum-overclassical advantage for information processing can be derived from the Kochen-Specker theorem alone.

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Recent theoretical and experimental work on quantum computation and quantum information theory has inspired renewed interest in fundamental results of quantum mechanics. The Horodeckis have shown, for example, that a spin-$\frac{1}{2}$ state can be teleported with greater than classical fidelity using any mixed two spin-$\frac{1}{2}$ state which violates some generalized Bell-CHSH inequality [1]. Quantum teleportation was first demonstrated experimentally in quantum optics systems [2]; the parametric down-conversion techniques crucial for these experiments have also been used to verify violation of Bell's inequality directly [3]. Although the Bell-CHSH inequalities were originally derived in the context of EPR-B experiments [4] and (local) hidden variable theories [5,6], the present concern is with the differences in information processing capability between classical and quantum systems [7].

Analyses of EPR-B experiments from the very first [8] have been concerned with limitations in, for example, detector efficiency [6]: The observed violations of Bell-CHSH inequalities are consequently reduced; so, too, is teleportation fidelity [2,9].

Logically, if not entirely chronologically prior contradictions with (noncontextual) hidden variable theories were derived by Bell [10] from a theorem of Gleason [11] and by Kochen and Specker [12]. The GHZ-Mermin three spin-$\frac{1}{2}$ state exhibits a similar incompatibility with (noncontextual) hidden variable theories [13] and reduces the communication complexity of certain problems [14]. While no quantum improvement in information processing power has yet been derived directly from the Kochen-Specker theorem, it is natural to ask for the consequences of experimental limitations on measurement in this setting.

The Kochen-Specker theorem concerns the results of a (counterfactual) set of measurements on a quantum system described by a vector in a three dimensional Hilbert space. Kochen and Specker consider, for example, measurement of the squares of the three angular momentum components of a spin-1 state [12]. The corresponding operators commute and can be measured simultaneously, providing one "yes" and two "no's" to the three questions, "Does the spin component along $\hat{a}$, $\hat{b}$, $\hat{a} \times \hat{b}$ vanish?" for any $\hat{a} \perp \hat{b} \in S^2$, the unit sphere in $\mathbb{R}^3$. Specker [15] and Bell [10] observed that Gleason's theorem [11] implies that there can be no assignment of "yes's" and "no's" to the vectors of S^2 consistent with this requirement:

each triad is "colored" with one "yes" and two "no's" (1)

(where *triad* means three mutually orthogonal vectors) and concluded that there could be no theory with hidden variables assigned independently of the measurement context.

A compactness argument [16] implies that there must be a *finite* set of triads for which there is no coloring

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satisfying (1). Kochen and Specker gave the first explicit construction of such a finite set [12]. Their construction requires 117 vectors; subsequently, examples with 33 [17] and 31 [18] vectors in S^2 have been constructed.

For our present purposes the details of these constructions are unimportant; all that matters is that the vectors forming the set of triads are precisely specified. But, as Birkhoff and von Neumann remark in their seminal study of the lattice of projections in quantum mechanics, "it seems best to assume that it is the *Lebesgue-measurable* subsets ... which correspond to experimental propositions, two subsets being identified, if their difference has *Lebesgue measure* 0" [19, p. 825]. That is, no experimental arrangement could be aligned to measure spin projections along coordinate axes specified with more than finite precision. The triads of a Kochen-Specker construction should therefore be constrained only to lie within some (small) neighborhoods of their ideal positions. This is sufficient to nullify the Kochen-Specker theorem because, as we will show presently, there is a coloring of the vectors of a set of triads, dense in the space of triads, which respects (1). More complicated colorings satisfying (1) "almost everywhere" have been constructed by Pitowsky using the axiom of choice and the continuum hypothesis [20]; our results here support a conjecture of his that many dense subsets—in particular, the rational vectors—have colorings which satisfy (1) [21], but we will need no more than constructive set theory.

The finite constructions violating (1) provide the clue we use: In each case the components of some of the vectors forming triads are irrational numbers. So let us consider only the vectors with rational components, $S^2 \cap \mathbb{Q}^3$. This is a familiar subset: The usual requirement of separability [22] for Hilbert space and for the lattice of measurement propositions depends upon such a countable dense subset [23]; the fact that it is dense means that it is indistinguishable from its closure by finite precision measurements. As Jauch puts it, while the rationals must already be defined with infinite precision, completing them to include the irrationals requires that "we transcend the proximably observable facts and ... introduce *ideal elements* into the description of physical systems" [23, p. 75]. Surely the meaning of quantum mechanics should not rest upon such nonexperimental entities. But, at least in the three dimensional arena for the Kochen-Specker theorem it does, as we will be able to conclude from the following three lemmas:

Lemma 1: *The rational vectors $S^2 \cap \mathbb{Q}^3$ can be colored to satisfy* (1).

Proof: This is an immediate consequence of a result of Godsil and Zaks [24] which is in turn based upon a theorem of Hales and Straus [25]. It suffices here to give an explicit coloring using their results. The rational projective plane $\mathbb{Q}P^2$ consists of triples of integers (x, y, z) with no common factor other than 1 (every integer is taken to divide 0). Since at least one of x, y, and z must therefore be odd, and since odd (even) numbers square to 1 (0) modulo 4, exactly one must be odd if $x^2 + y^2 + z^2$ is to be a square. In this case, and only in this case, $(x, y, z) \in \mathbb{Q}P^2$ can be identified as a vector in $S^2 \cap \mathbb{Q}^3$. Consider a triad of such vectors. For any two, (x, y, z) and (x', y', z'), $x'x + y'y + z'z = 0$ implies that they must differ in which component is odd. Thus exactly one vector of any triad has an odd z component. Color this one "yes" and the other two "no." This defines a *z-parity* coloring of $S^2 \cap \mathbb{Q}^3$ satisfying (1).

The rational vectors are dense in S^2 since $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$ and rational vectors in S^2 map bijectively to rational points in the affine plane—since stereographic projection is a birational map. Furthermore,

Lemma 2: *The rational vectors z-parity colored "yes" are dense in S^2.*

Proof: Again we follow the argument of Godsil and Zaks [24]: Rotation by angle $\alpha = \arccos\frac{3}{5}$ about the x axis takes each rational vector $(0, y, z)$ with odd z (i.e., colored "yes") to another rational vector colored "yes." Since α is not a rational multiple of π, iterated rotation takes $(0, 0, 1)$ to a dense set of vectors in the $x = 0$ great circle of S^2. Similarly, iterated rotation by angle α around the z axis takes this set of vectors dense in S^1 to a set of vectors dense in S^2, each of which is colored "yes" since it has an odd z component. ■

Repeating this argument with (x, y, z) permuted to (y, z, x) shows that the rational vectors z parity colored "no" are also dense in S^2.

Lemma 3: *The rational triads are dense in the space of triads.*

Proof: By the proof of Lemma 2, for any $\epsilon > 0$, within a $\frac{1}{2}\epsilon$-neighborhood of a specified vector $\hat{a}$ of a triad, $\hat{a}$, $\hat{b}$, $\hat{a} \times \hat{b}$, there is a rational vector $\hat{u}$ to which $(0, 0, 1)$ is mapped by an $SO(3, \mathbb{Q})$ rotation. This rotation maps the rational vectors $(x, y, 0)$ on the equator to the rational vectors in a great circle passing through the $\frac{1}{2}\epsilon$-neighborhoods of $\hat{b}$ and $\hat{a} \times \hat{b}$. Since the rational points are dense in the equator (also a consequence of the proof of Lemma 2) there is a rational vector $\hat{v} \perp \hat{u}$ in the $\frac{1}{2}\epsilon$-neighborhood of $\hat{b}$, and thus $\hat{u} \times \hat{v}$ is a rational vector in the ϵ-neighborhood of $\hat{a} \times \hat{b}$. ■

Suppose we measure some triad in a three dimensional Kochen-Specker construction. By Lemma 3 the unavoidable finite precision of this measurement cannot distinguish it from the (many) rational triads within some neighborhood of the intended triad. By Lemmas 1 and 2 the results of a (counterfactual) set of such measurements cannot conflict with (1) and so cannot rule out a noncontextual hidden variable theory *defined over the rationals*. Thus finite precision measurement nullifies the Kochen-Specker theorem. The z-parity coloring of $S^2 \cap \mathbb{Q}^3$ shows that arguments such as Bell's [10], based on Gleason's theorem [11] in three dimensions, also fail when the finite precision of measurement is taken into account [26].

Although our explicit construction involves the rational vectors, we emphasize that they are incidental to the interpretation of this result. *Any* dense subset is indistinguishable by finite precision measurement from its completion, so any colorable dense subset would be equally well. Our results, together with Pitowsky's earlier [20] and Kent's subsequent [26] constructions indicate that there are many such subsets.

We conclude by remarking that while one might object that since the counterfactual measurements specified by a Kochen-Specker construction are not (simultaneously) experimentally realizable, it is unreasonable to impose the experimental limitation of finite precision on such a theoretical edifice. But theoretical analyses of the power of algorithms must address the possibility that it resides in infinite precision specification of the computational states or the operations on them. Schönhage showed, for example, that classical computation with infinite precision real numbers would solve *NP*-complete problems efficiently [27]. And, as Freedman has emphasized, even classical statistical mechanics models would solve #P-hard problems were infinite precision measurement possible [28]. The promise of *quantum* computation, in contrast, is efficient algorithms—which require only poly(log) number of bits precision—for problems not known to have polynomial time classical solutions [29]. Thus, despite the relation noted earlier with the GHZ-Mermin state which can reduce communication complexity, the elementary argument presented here shows that given the finite precision of any experimental measurement, the Kochen-Specker theorem alone cannot separate quantum from classical information processing in three dimensional Hilbert space. We have not, of course, constructed even a static (much less a dynamic) hidden variable theory for a spin-1 particle, so we have failed to prove that no separation result is possible—only that the Kochen-Specker theorem does not imply one, as we might have expected. Our results, and Pitowsky's deterministic model [20], however, make it seem unlikely that any separation exists [30].

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