Application of Information Theory in Nuclear Liquid Gas Phase Transition

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Information entropy and Zpif's law in the field of information theory have been used for studying the disassembly of nuclei in the framework of the isospin dependent lattice gas model and the molecular dynamical model. We found that the information entropy in the event space is maximum at the phase transition point and the mass of the cluster shows exactly inversely to its rank, i.e., Zpif's law appears. Both novel criteria are useful in searching the nuclear liquid gas phase transition experimentally and theoretically.

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Hot nuclei can be formed in energetic heavy ion collisions (HIC) and may be highly excited. They deexcite by different decay modes, such as multifragmentation. Experimentally, this kind of multifragment emission was observed to evolve with beam energy (excitation energy, or nuclear temperature, ...). Multiplicity, N_{imf}, of intermediate mass fragment (IMF) rises with the beam energy, reaches a maximum, and finally falls to lower value. This phenomenon of the rise and fall of N_{imf} may be related to the liquid gas phase transition in nuclear matter [1]. The onset of multifragmentation probably indicates the coexistence of liquid and gas phases. The mass distribution of IMF distribution can be expressed as a power law with parameter τ. The minimum of τ, $\tau_{\min}$, occurs when the liquid gas phase transition takes place [2]. However, $\tau_{\min}$ can also reveal at supercritical densities along the Kertész line [3] and at some subcritical densities at lower temperature [4]. So it is not possible to determine the phase transition only from N_{imf} and $\tau_{\min}$.

On the other hand, experimentalists measured the nuclear caloric curves, i.e., the relationship between nuclear temperature and the excitation energy. He-Li isotopic temperature from the Albergo thermometer [5] for projectile-like Au spectators seems to exhibit a temperature plateau in the excitation energy range of 3 to 10 MeV/u [6]. This plateau was taken as an indication for a first order nuclear liquid gas phase transition. However, due to the changing mass of Au spectators with excitation energy and the side-feeding effect to measured He-Li isotopic temperature, this conclusion is questionable [7]. Nuclear caloric curves were also surveyed by several groups [8]. However, unfortunately, the sharp signature of the liquid gas phase transition in macroscopic systems may be smoothed and blurred due to the small numbers of nucleons in nuclei, and/or the difficulty to perform a direct comparison between the measured "apparent" temperature and the "real" temperature interferes in obtaining the real nuclear caloric curve. These factors hamper the reaching of a definite conclusion on liquid gas phase transition in nuclei.

The extraction of critical exponents and the study of critical behavior in finite-size systems were attempted in [9] and were followed by controversial debates [10]. In this context, it is necessary and meaningful to search for some novel signatures to characterize the nuclear liquid gas phase transition in order to guide the experimental analysis and theoretical predictions.

In this Letter, we will introduce information entropy [11], H, and Zpif's law [12] into the diagnosis of a nuclear liquid gas phase transition. The information entropy was defined by Shannon in information theory. Originally, it measures the "amount of information" which is contained in messages sent along a transmission line. It can be expressed as follows:

$$H = -\sum_i p_i \ln(p_i), \tag{1}$$

where p_i is a normalized probability, and $\sum_i p_i = 1$. Jaynes proposed that a very general technique for discovering the least biased distribution of the p_i consists in the maximization of the Shannon H entropy, subject to whatever constraints on p_i are appropriate to the particular situation. The maximization of H was thus put forward as a general principle of statistical inference—one which could be applied to a wide variety of problems in economics, engineering, and many other fields, such as quantum phenomena [11]. In high energy hadron collisions, multiparticle production proceeds on the maximum stochasticity, i.e., they should obey the maximum entropy principle. This kind of stochasticity can be also quantified via the information entropy which has been shown to be a good tool to measure chaoticity in the hadron decaying branching process [13]. In different physical conditions, information entropy can be expressed with different stochastic variables. In this work on HIC, we define p_i as the event probability of having "i" particles produced, i.e., $\{p_i\}$ is the normalized probability distribution of total multiplicity, and the sum is taken over whole $\{p_i\}$ distribution. This emphasis is on the event space rather than the phase space. As shown below, this kind of information entropy [14] can be taken as a method to determine nuclear liquid gas phase transition.

Zpif's law [12] has been known as a statistical phenomenon concerning the relation between English words and their frequency in literature in the field of linguistics. The law states that, when we list the words in the order of decreasing population, the frequency of a word is inversely proportional to its rank [12]. This relation was found not only in linguistics but also in other fields of sciences. For instance, the law appeared in distributions of populations in cities, distributions of income of corporations, distributions of areas of lakes, and cluster-size distribution in the percolation process [15]. In this Letter, Zpif's law will be tested for the fragment mass distribution and it is evidenced to be a factor in characterizing the phase transition.

The tools we will use here are the isospin dependent lattice gas model (LGM) and the molecular dynamical (MD) model. The lattice gas model was developed to describe the liquid gas phase transition for an atomic system by Lee and Yang [16]. The same model has already been applied to nuclear physics for isospin symmetrical systems in the grand canonical ensemble [17] with a sampling of the canonical ensemble [3,4,18–22], and also for isospin asymmetrical nuclear matter in the mean field approximation [23]. In addition, a classical molecular dynamical model is used to compare its results with the results of the lattice gas model. Here we will make a brief description for the models.

In the lattice gas model, A $(= N + Z)$ nucleons with an occupation number s, which is defined $s = 1$ (-1) for a proton (neutron) or $s = 0$ for a vacancy, are placed on the L sites of lattice. Nucleons in the nearest neighboring sites interact with an energy $\epsilon_{s_i s_j}$. The Hamiltonian is written as $E = \sum_{i=1}^{A}(P_i^2/2m) - \sum_{i<j}\epsilon_{s_i s_j} s_i s_j$. The interaction constant $\epsilon_{s_i s_j}$ is chosen to be isospin dependent and be fixed to reproduce the binding energy of the nuclei [20]:

$$\begin{aligned}\epsilon_{nn} &= \epsilon_{pp} = 0\ \text{MeV}\,,\\ \epsilon_{pn} &= -5.33\ \text{MeV}\,.\end{aligned} \tag{2}$$

A three-dimensional cubic lattice with L sites is used. The freeze-out density of the disassembling system is assumed to be $\rho_f = \frac{A}{L}\rho_0$, where ρ_0 is the normal nuclear density. The disassembly of the system is to be calculated at ρ_f, beyond which nucleons are too far apart to interact. Nucleons are put into a lattice by Monte Carlo Metropolis sampling. Once the nucleons have been placed, we also ascribe to each of them a momentum by Monte Carlo samplings of Maxwell-Boltzmann distribution.

Once this is done, the LGM immediately gives the cluster distribution using the rule that two nucleons are part of the same cluster if $P_r^2/2\mu - \epsilon_{s_i s_j} s_i s_j < 0$. This method is similar to the Coniglio-Klein's prescription [24] in condensed matter physics and was shown to be valid in LGM [3,4,19,21]. To calculate clusters using MD, we propagate the particles from the initial configuration for a long time under the influence of the chosen force. The form of the force is chosen to be also isospin dependent in order to compare with the results of LGM. The potential for unlike nucleons is

$$\begin{aligned}v_{\rm np}(r)\,(r/r0 < a) &= C[B(r_0/r)^p - (r_0/r)^q]\\ &\quad\times \exp\left(\frac{1}{(r/r_0) - a}\right), \\ v_{\rm np}(r)\,(r/r_0 > a) &= 0\,,\end{aligned} \tag{3}$$

where $r_0 = 1.842$ fm is the distance between the centers of two adjacent cubes. The parameters of the potentials are $p = 2$, $q = 1$, $a = 1.3$, $B = 0.924$, and $C = 1966$ MeV. With these parameters, the potential is minimum at r_0 with the value -5.33 MeV, is zero when the nucleons are more than $1.3r_0$ apart, and becomes strongly repulsive when r is significantly less than r_0. The potential for like nucleons is written as

$$\begin{aligned}v_{\rm pp}(r)\,(r < r0) &= v_{\rm np}(r) - v_{\rm np}(r_0)\,,\\ v_{\rm pp}(r)\,(r > r0) &= 0\,.\end{aligned} \tag{4}$$

The system evolves with the above potential. At asymptotic times the clusters are easily recognized. Observables based on the cluster distribution in both models can now be compared. In the case of proton-proton interactions, the Coulomb interaction can also be added separately, and it can be compared with the case without Coulomb effects.

In this Letter, we choose the medium size nuclei ^{129}Xe as an example. In most cases, ρ_f is chosen to be $0.38\rho_0$, since the experimental data can be best fitted by ρ_f between $0.3\rho_0$ and $0.4\rho_0$ in previous LGM calculations [19,25], which corresponds to 7^3 cubic lattice. In addition, $0.18\rho_0$, corresponding to 9^3 cubic lattice and $0.60\rho_0$, corresponding to 6^3 cubic lattice of ρ_f, are also taken to compare and check the results with different ρ_f values in the LGM case. 1000 events were simulated for each combination of T and ρ_f which ensures enough statistics for results.

In order to check the phase transition behavior in LGM and MD, we will first show the results of some physical observables, namely, the effective power-law parameter τ, the second moment of the cluster distribution S_2, and the multiplicity of intermediate mass fragments $N_{\rm imf}$, for the disassembly of ^{129}Xe (Fig. 1). These observables were shown to be good indicators of a liquid gas phase transition, as shown in Ref. [20]. The extreme values of τ, $N_{\rm imf}$, and S_2 occur at the same temperature, indicating the onset of the phase transition, for each calculation case. For the LGM case, the phase transition temperature increases with the freeze-out density; for the MD case, a slight small transition temperature is obtained when Coulomb force is ignored. It becomes much lower in the case of Coulomb interaction due to its long range repulsion. A similar phenomenon has been explored in a previous study [20]. However, the aim of this Letter is to take the above transition temperatures as references to search for novel signatures of liquid gas phase transition.

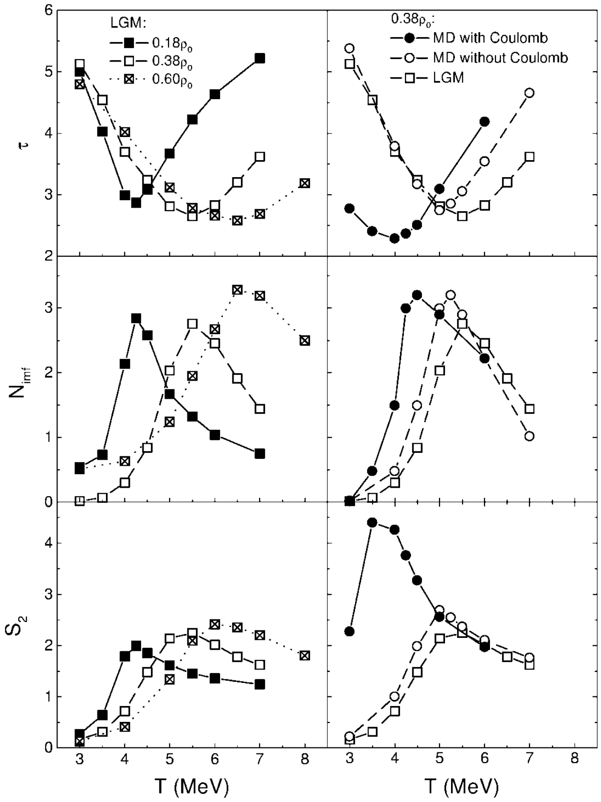

FIG. 1. The effective power-law parameter τ, the second moment of the cluster distribution S_2, and the multiplicity of intermediate mass fragments N_{imf}, as a function of temperature for the disassembly of ^{129}Xe. Left panel is the LGM results with different ρ_f and right panel is the comparison of MD to LGM with $0.38\rho_0$. The symbols are illustrated in the figure.

Figure 2 shows the information entropy for disassembly of Xe. The information entropy exhibits a rise and fall with temperature, which is similar to the behaviors of N_{imf} and S_2. The temperatures extracted from the peak values of H are consistent with the transition temperatures in Fig. 1, indicating that information entropy ought to be a good diagnosis of phase transition. Physically, the maximum of H reflects the largest fluctuation of the multiplicity probability distribution in the phase transition point. In this case, it is the most difficult to predict how many clusters will be produced in each event, i.e., the disorder (entropy) of information is the largest. Generally speaking, the larger the dispersal of multiplicity probability distribution, the higher the information entropy, and then the disorder of the system in the event topology. One should make a careful distinction between this information entropy, on the one hand, and the original thermodynamic entropy, on the other hand [11,14]. The latter generally illustrates the heat disorder in momentum space rather than event space and it always increases with temperature.

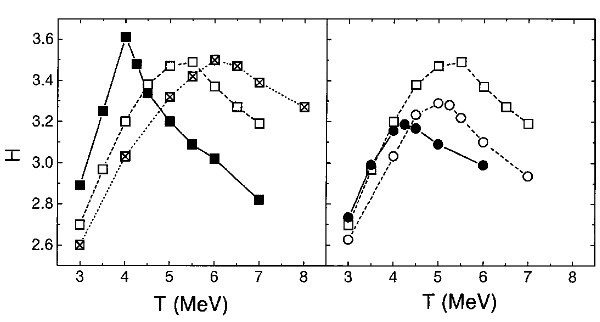

FIG. 2. The same as Fig. 1, but for the information entropy H.

Next, we will present the results for testing Zpif's law in the fragment distribution. The law states that the relation between the sizes and their ranks is described by $A_n = c/n$ $(n = 1, 2, 3, \ldots)$, where c is a constant, and A_n is the mass of rank n in a mass list when we arrange the clusters in the order of decreasing size. In the calculations, we averaged the masses for each rank in mass lists of the events. Then we examined the relation between the masses A_n and their ranks n with the fit of $A_n \propto n^{-\lambda}$ in the range of $1 \leq n \leq 10$, where λ is the slope parameter. The upper panel of Fig. 3 summarizes such parameter λ as a function of temperature for LGM and MD cases. Clearly, the value of λ decreases with temperature, indicating that the difference of mass between the different fragment ranks is becoming smaller. When $\lambda \sim 1$, the Zpif's law is satisfied: $A_n \propto n^{-1}$. The temperatures having Zpif's law for all calculations in Fig. 3 are also consistent with the respective transition temperatures extracted from the extreme values of some observables in Figs. 1 and 2. Therefore Zpif's law is also a good signal to phase transition. From the statistical point of view, Zpif's law is related to the critical behavior or self-organized criticality [2,26], which may be a special state with the maximum information or least effort.

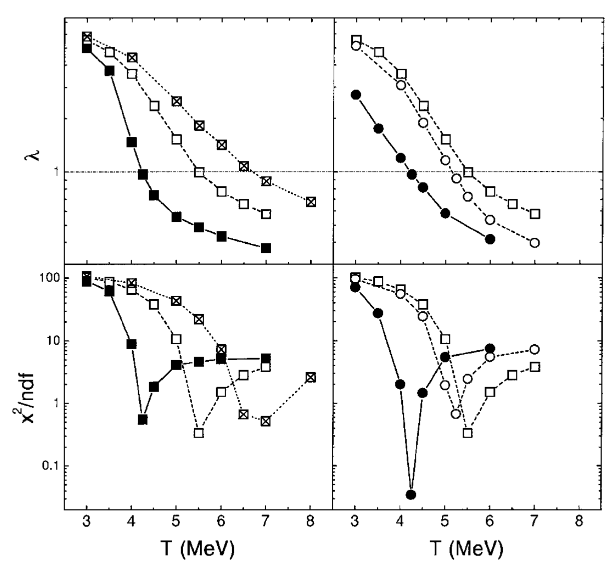

FIG. 3. The slope parameter λ of the relation A_n to n (upper panel) and the χ^2/ndf with the fit of Zpif's law (lower panel) as a function of temperature. Left panel is the LGM results with different ρ_f and right panel is the comparison of MD to LGM with $0.38\rho_0$. The symbols are the same as Fig. 1.

In order to further illustrate that Zpif's law exists at the phase transition point most probably, we directly fit the rank-classified cluster distribution with Zpif's law and extract the truth of the hypothesis: χ^2 test. The lower panel of Fig. 3 demonstrates the χ^2/ndf for the A_n-n relations at different T for different calculation cases. As expected, there are minima of χ^2/ndf around the respective transition temperature, which further support Zpif's law of the fragment distribution indicating a phase transition. All calculations give the same conclusions as above.

In addition, we also investigated larger systems, such as $A = 274$, 500, and 830, in the LGM case to see if the system behaves as expected in ^{129}Xe. The results show that the maximum of information entropy and Zpif's law behavior still remain at the same phase transition temperature as the one extracted from the extreme values of τ, $N_{\rm imf}$, and S_2. It illustrates that both novel criteria are suitable as signals of phase transition in the larger A limit.

In conclusion, the information entropy and Zpif's law in the field of information theory are introduced, for the first time, into the study of the liquid gas phase transition of nuclei in the framework of the isospin dependent lattice gas model and the molecular dynamical model in a ρ-T plane. At the point of phase transition, the information entropy of multiplicity distribution is maximum, which indicates that the system at this time has the largest fluctuation/stochasticity/chaoticity in the event space. Meanwhile, the cluster mass shows exactly inversely to its rank, i.e., Zpif's law appears. Even though both criteria are still phenomenological, we believe that they are simple and practicable tools to diagnose the nuclear liquid gas phase transition in experiments and theories. We are waiting for some data analysis on the information entropy and Zpif's law in the near future.

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