Spacetime and the Holographic Renormalization Group

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Anti–de Sitter space can be foliated by a family of nested surfaces homeomorphic to the boundary of the space. We propose a holographic correspondence between theories living on each surface in the foliation and quantum gravity in the enclosed volume. The flow of observables between our "interior" theories is described by a renormalization group equation. The dependence of these flows on the foliation of space encodes bulk geometry.

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The holographic principle [1] states that quantum gravity on a manifold can be described by a theory defined on the boundary of that manifold. The simplest realization of this principle has been in Anti–de Sitter (AdS) space, which, in certain cases, can be described by a *local* conformal field theory (CFT) defined on the AdS boundary [2]. The correlation functions of the CFT describe the experiments of an observer who prepares field configurations at infinity and measures their amplitudes.

A strong version of the holographic principle would assert that quantum gravity on any volume contained within a manifold can be described by a theory defined on the boundary of that volume. The holographic dual would then describe experiments of an observer who prepares field configurations on the interior boundary and measures their amplitudes.

In this Letter, we address the issue of interior holographic duals for AdS by adopting a Wilsonian renormalization group (RG) perspective. To describe a subset of a system we "integrate out" the excluded degrees of freedom. In general, this will induce an infinite set of interactions in the remaining theory, making it nonlocal. In the AdS context, we foliate spacetime by surfaces $\partial\mathcal{M}_\rho$ of constant radial coordinate ρ, with enclosed volume $\mathcal{M}_\rho$. We fix the values of the fields Φ_ρ on $\partial\mathcal{M}_\rho$ and perform the bulk path integral over the excluded volume. The result is a nonlocal functional of Φ_ρ which we treat as a boundary contribution to the bulk action describing $\mathcal{M}_\rho$. Responses of the resulting interior path integral to variations of Φ_ρ describe experiments carried out by observers placed on $\partial\mathcal{M}_\rho$. We identify these responses with the correlation functions of a holographic dual defined on the interior boundary. Related work has appeared recently in [3]. For some other discussions of RG equations in the AdS and CFT context, see [4].

The observer at $\partial\mathcal{M}_\rho$ naturally probes the interior volume with pointlike variations of the fields Φ_ρ. In the semiclassical limit, the bulk equations of motion tell us that these variations turn into extended variations of the fields at infinity (see, e.g., [5] and references therein). This spreading of the fields increases as $\partial\mathcal{M}_\rho$ is moved into the interior. In the CFT dual, these boundary values of bulk fields map onto sources smeared over a characteristic scale specified by the position of the inner boundary. It is then appropriate to integrate out CFT degrees of freedom at lengths shorter than this scale. This suggests that the interior holographic theories described above are related to the CFT duals of AdS spaces by coarsening transformations. We will demonstrate that this is the case and show that, for any nested family of foliating surfaces for AdS, there is an RG equation describing the flow of observables in the corresponding series of interior holographic duals. Spacetime diffeomorphisms relate foliating families and are realized as relations between different flows.

(1) Defining the inner correspondence.—We will consider Euclidean AdS, which is topologically a ball. Foliate AdS by a family of topologically spherical surfaces indexed by a parameter ρ approaching 0 at the boundary and ∞ at the center. Let $\partial\mathcal{M}_\rho$ be any element of this foliating family, with $\mathcal{M}_\rho$ being the enclosed volume. The AdS/CFT correspondence for the boundary at $\rho = 0$ is written as [6,7]

$$e^{-Z_0[\Phi_0]} = \int_{\mathcal{M}_0} \mathcal{D}\Phi\, e^{-S_0[\Phi]} = \langle e^{-\int_{\partial\mathcal{M}_0}\Phi_0\mathcal{O}}\rangle = e^{-S_{\mathrm{CFT}}(\Phi_0)}. \tag{1}$$

The two terms on the left represent the string theory path integral on AdS evaluated as a functional of the boundary data Φ_0. On the right-hand side is the effective action for the dual conformal field theory with sources Φ_0. The spacetime action S_0 contains both bulk and boundary contributions: $S_0[\Phi] = \int_{\mathcal{M}_0}\mathcal{L}[\Phi] + \int_{\partial\mathcal{M}_0} B_0[\Phi_0]$, where the boundary terms are chosen to cancel divergences arising from the bulk integral (see, e.g., [8]). Upon performing the bulk path integral, $Z_0[\Phi_0]$ becomes a functional of Φ_0 defined on $\partial\mathcal{M}_0$. Since the conformal factor on the boundary of AdS actually diverges, it is convenient to cut off the space at some small $\rho = \epsilon$, which can be understood as a kind of ultraviolet regulator for the CFT

[9]. (We will always take $\epsilon \to 0$ in the end.) We write

$$Z_\epsilon[\Phi_\epsilon] = \sum_{n=1}^{\infty} \int_{\partial \mathcal{M}_\epsilon} \left[\prod_{j=1}^{n} d\mathbf{b}_j \sqrt{\gamma_\epsilon(\mathbf{b}_j)}\, \Phi_\epsilon(\mathbf{b}_j) \right] \times c_n(\epsilon; \mathbf{b}_1, \ldots, \mathbf{b}_n). \quad (2)$$

Here $\mathbf{b}$ are boundary coordinates and γ_ϵ is the determinant of the induced metric on $\partial \mathcal{M}_\epsilon$. The correlation functions of the dual CFT are precisely the coefficients c_n in the $\epsilon \to 0$ limit.

We are interested in defining a suitable *inner correspondence* between quantum gravity on $\mathcal{M}_\rho$ and some theory defined on the boundary $\partial \mathcal{M}_\rho$. In the field theory limit we would like an equation analogous to (1):

$$e^{-Z_\rho[\Phi_\rho]} = \int_{\mathcal{M}_\rho} \mathcal{D}\Phi\, e^{-S_\rho[\Phi]} = e^{-S_{\mathrm{CFT}}(\Phi_\rho)}. \quad (3)$$

Consider an observer stationed on $\partial \mathcal{M}_\rho$. Such an observer can probe physics in the region $\mathcal{M}_\rho$ by measuring the amplitudes for various field configurations Φ_ρ to occur. The amplitudes are given by the path integral in the full AdS spacetime subject to the boundary condition that $\Phi = \Phi_\rho$ on $\partial \mathcal{M}_\rho$. It is convenient to perform the path integral in two steps. First, integrate over fields in the excluded volume $\mathcal{M}_0 - \mathcal{M}_\rho$ to get a nonlocal functional of Φ_ρ: $e^{-Z_\rho[\Phi_\rho]} = \int_{\mathcal{M}_\rho} \mathcal{D}\Phi \int_{\mathcal{M}_0 - \mathcal{M}_\rho} \mathcal{D}\Phi\, e^{-S_0[\Phi]} = \int_{\mathcal{M}_\rho} \mathcal{D}\Phi\, e^{-S_\rho[\Phi]}$. $S_\rho[\Phi]$ encapsulates the physics in $\mathcal{M}_\rho$. The virtue of first integrating over the bulk fields in the excluded volume is that we can envision doing the analogous procedure in the gauge theory. Roughly speaking, fields Φ_ρ correspond to smeared fields Φ_0 at the outer boundary, and hence to smeared sources in the gauge theory. In the CFT it is then natural to form an effective action by integrating over field modes with wavelengths shorter than the smearing length.

To compute bulk correlation functions on $\partial \mathcal{M}_\rho$ we perform the remaining path integral over $\mathcal{M}_\rho$ to obtain a functional of Φ_ρ,

$$Z_\rho[\Phi_\rho] = \sum_{n=1}^{\infty} \int_{\partial \mathcal{M}_\rho} \left[\prod_{j=1}^{n} d\mathbf{b}_j \sqrt{\gamma_\rho(\mathbf{b}_j)}\, \Phi_\rho(\mathbf{b}_j) \right] \times c_n(\rho; \mathbf{b}_1, \ldots, \mathbf{b}_n). \quad (4)$$

We have obtained a one parameter set of correlation functions $c_n(\rho; \mathbf{b}_1, \ldots, \mathbf{b}_n)$ indexed by ρ which, by construction, reduce to those in (2) as $\rho \to \epsilon$. The dependence on ρ is naturally interpreted as the renormalization group evolution of the correlation functions.

Semiclassical correspondence: In the semiclassical, small curvature, limit the bulk path integral for the "outer correspondence" (1) is dominated by its saddle points. So, in the corresponding limit of the dual CFT, Eq. (1) becomes $e^{-S_{\mathrm{cl}}(\Phi_0)} = e^{-S_{\mathrm{CFT}}(\Phi_0)}$. The left-hand side is now simply the AdS classical action evaluated as a functional of boundary data. To define the "inner correspondence" in the field theory limit we simply integrated over the fields in the excluded volume $\mathcal{M}_0 - \mathcal{M}_\rho$. In the semiclassical limit this amounts to evaluating the action for a classical solution in the excluded volume with fields taking values Φ_ρ at the inner boundary. There is a unique solution in the bulk with the prescribed boundary conditions, at least in perturbation around the free limit.

We can compute the full bulk action associated with classical solutions and express it in terms of either the fields Φ_ϵ or Φ_ρ: $S_{\mathrm{cl}} = \sum_{n=1}^{\infty} \int_{\partial \mathcal{M}_\epsilon} [\prod_{j=1}^{n} d\mathbf{b}_j \times \sqrt{\gamma_\epsilon(\mathbf{b}_j)}\, \Phi_\epsilon(\mathbf{b}_j)] c_n(\epsilon; \mathbf{b}_1, \ldots, \mathbf{b}_n) = \sum_{n=1}^{\infty} \int_{\partial \mathcal{M}_\rho} [\prod_{j=1}^{n} \times d\mathbf{b}_j \sqrt{\gamma_\rho(\mathbf{b}_j)}\, \Phi_\rho(\mathbf{b}_j)]\, c_n(\rho; \mathbf{b}_1, \ldots, \mathbf{b}_n)$. To derive an RG equation we must relate the correlation functions at ρ to those at ϵ. Such a relation is found by noting, as above, that the classical fields Φ_ϵ are uniquely specified by Φ_ρ. We display this relation in terms of a propagator,

$$\Phi_\epsilon(\mathbf{b}) = \int_{\partial \mathcal{M}_\rho} \sqrt{\gamma_\rho(\mathbf{b}')}\, G_{\epsilon\rho}(\mathbf{b}, \mathbf{b}')\, \Phi_\rho(\mathbf{b}'). \quad (5)$$

The meaning of our construction: The meaning of our construction is most easily grasped by considering two point functions in the inner and outer theories. In the semiclassical limit, the outer CFT two point function for widely separated operators is computed from a classical bulk geodesic between two boundary points. Our procedure for computing inner two point functions amounts to extending the geodesic between two interior points until they reach the outer boundary, and adding in the action for the excluded part of the trajectory. Since the geodesics spread on the way from the interior boundary to the exterior, interior correlators at one separation are given by exterior correlators at a larger separation. More concretely, consider AdS in Poincaré coordinates: $ds^2 = \frac{\ell^2}{\rho^2}(d\rho^2 + d\mathbf{b}^2)$. Consider a scalar field in AdS in a representation of the conformal group with weight Δ. Disturbances of this field on the AdS boundary ($\rho = 0$) propagate to the surface at fixed ρ via the kernel $G_{bb} \sim \frac{\rho^\Delta}{(\rho^2 + |\mathbf{b} - \mathbf{b}'|^2)^\Delta}$. So a point disturbance at $\partial \mathcal{M}_0$ grows to a coordinate size ρ at $\partial \mathcal{M}_\rho$. Conversely, a given point on $\partial \mathcal{M}_\rho$ is affected by fields within a patch of coordinate size ρ on the outer boundary. Now imagine an observer on $\partial \mathcal{M}_\rho$ who probes the system with local sources. In terms of the original CFT, such an observer has access only to sources which are smeared over coordinate size ρ. So her experiments can be reproduced by an effective action in which degrees of freedom smaller than ρ have been integrated out—short distance information has been lost. (If the observer can place sources on $\partial \mathcal{M}_\rho$ with arbitrary precision, all CFT degrees of freedom must be retained. This is because the effect of the smearing can be undone by making experiments at infinitesimal separations on $\partial \mathcal{M}_\rho$. But, of course, such infinite precision experiments are beyond the validity of our supergravity analysis.) Integrating out degrees of freedom induces an infinite series of higher derivative terms, multiplied by powers of the dimensionful scale ℓ. If one tries to pass to the flat space limit by sending $\ell \to \infty$, the coefficients of the higher derivative terms diverge, signaling an increasingly nonlocal description.

(2) RG flow of observables.—We will now show that the flow of observables between our "inner" theories is described by a renormalization group equation. As before, foliate Euclidean AdS by a family of surfaces homeomorphic to the boundary, and let n^μ be the outward pointing normal to this family of surfaces. Then, if the spacetime metric is $g_{\mu\nu}$, the induced metric on a given foliating surface is $\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$. In an adapted coordinate system, with ρ being the radial direction, the metric admits an Arnowitt-Deser-Misner-like decomposition: $g_{\mu\nu} = g_{\rho\rho}\, d\rho^2 + \gamma_{ij}\,(db^i + V^i\, d\rho)\,(db^j + V^j\, d\rho)$, $n^\mu = \delta^{\mu\rho}/\sqrt{g_{\rho\rho}}$. Using (5) we find that

$$c_n(\rho;\mathbf{b}_1,\ldots,\mathbf{b}_n) = \int_{\partial\mathcal{M}_\epsilon}\left[\prod_{j=1}^{n} d\mathbf{b}'_j\,\sqrt{\gamma_\epsilon(\mathbf{b}'_j)}\,G_{\epsilon\rho}(\mathbf{b}'_j,\mathbf{b}_j)\right] \times c_n(\epsilon;\mathbf{b}'_1,\ldots,\mathbf{b}'_n)\,. \qquad (6)$$

We have just learned that the observables of the inner theory are precisely the "outer" CFT correlators convolved against the kernel $G_{\epsilon\rho}$. To make progress, consider situations where we can undo the convolution by an integral transform. For example, if the metric on $\partial\mathcal{M}_\epsilon$ is proportional to the identity, the Fourier transform converts the convolution into a product. We will therefore refer to c_n in the deconvolved basis as the "momentum space" correlator $\tilde{c}_n$, $c_n(\rho;\mathbf{k}_1,\ldots,\mathbf{k}_n) = \tilde{G}_{\epsilon\rho}(\mathbf{k}_1)\ldots\tilde{G}_{\epsilon\rho}(\mathbf{k}_n) \times \tilde{c}_n(\epsilon;\mathbf{k}_1,\ldots,\mathbf{k}_n)$. Here the variables $\mathbf{k}$ parametrize the deconvolution basis. The correlator $\tilde{c}_n(\epsilon;\ldots)$ is independent of the index ρ of the interior surface. So the ρ dependence of the inner observables is summarized by

$$\rho\,\frac{\partial}{\partial\rho}\,c_n(\rho;\mathbf{k}_1,\ldots,\mathbf{k}_n) + \left[\sum_j \rho\,\frac{\partial}{\partial\rho}\,\ln\tilde{G}_{\rho\epsilon}(\mathbf{k}_j)\right]\times c_n(\rho;\mathbf{k}_1,\ldots,\mathbf{k}_n) = 0\,. \qquad (7)$$

Equation (7) is an RG equation describing Wilsonian flow of correlators in the gauge theory, in correspondence with the observations of spacetime observers stationed on the fixed surfaces $\partial\mathcal{M}_\rho$.

Example: Poincaré coordinates: In Poincaré coordinates the metric of AdS is $ds^2 = \frac{\ell^2}{\rho^2}(d\rho^2 + d\mathbf{b}^2)$, and we are interested in surfaces of fixed ρ. We will work out the relation between inner and outer observables for massive scalars. In AdS_{d+1} the operator dual to such a scalar has dimension Δ, where $\Delta = \frac{d}{2} + \nu$, $\nu = \frac{1}{2}\sqrt{d^2 + 4m^2}$. To Fourier transform both sides of (6) it is convenient to define the inner and outer correlators in momentum space:

$$\tilde{c}_n(\rho;\mathbf{k}_1,\ldots,\mathbf{k}_n) = \int_{\partial\mathcal{M}_\rho}\left[\prod_{j=1}^{n} d\mathbf{b}_j\,\sqrt{\gamma_\rho(\mathbf{b}_j)}\,e^{i\mathbf{k}_j\cdot\mathbf{b}_j}\right] \times c_n(\rho;\mathbf{b}_1,\ldots,\mathbf{b}_n)\,. \qquad (8)$$

Next, since the propagator $G_{\epsilon\rho}$ approaches $\delta(\mathbf{b}-\mathbf{b}')/\sqrt{\gamma_\rho(\mathbf{b})}$ as $\epsilon \to \rho$, the Fourier transform with respect to $\mathbf{b}'$ gives $\tilde{G}_{\rho\rho}(\mathbf{b},\mathbf{k}) = e^{i\mathbf{k}\cdot\mathbf{b}}$. It is easy to construct a massive scalar mode solution that approaches such a plane wave on $\partial\mathcal{M}_\rho$ from the complete bases provided in, e.g., Ref. [10]. The propagator is then a Bessel function, $\tilde{G}_{\epsilon\rho}(\mathbf{b},\mathbf{k}) = (\epsilon/\rho)^{d/2}[K_\nu(q\epsilon)/K_\nu(q\rho)]e^{i\mathbf{k}\cdot\mathbf{b}}$ with $q^2 = \mathbf{k}\cdot\mathbf{k}$. Using the above propagator, we express the ρ dependence of $\tilde{c}_n(\rho;\mathbf{k}_1,\ldots,\mathbf{k}_n)$ in terms of Bessel functions, and then employ a series expansion to obtain

$$\tilde{c}_n(\rho;\mathbf{k}_1,\ldots,\mathbf{k}_n) = \left(\frac{\rho}{\epsilon}\right)^{n(\Delta-d)}\left[\prod_j \frac{1 + F(q_j^2\epsilon^2)}{1 + F(q_j^2\rho^2)}\right] \times \tilde{c}_n(\epsilon;\mathbf{k}_1,\ldots,\mathbf{k}_n)\,, \qquad (9)$$

where $F(z^2) = \sum_{n=1}^{\infty} e_{-\nu}(n)\,z^{2n} - z^{2\nu}\sum_{n=0}^{\infty} e_\nu(n)\,z^{2n}$. We implicitly understood all along that $\epsilon \to 0$. Since the theory is conformally invariant, this limit yields the scaling behavior $\tilde{c}_n(\epsilon;\mathbf{k}_1,\ldots,\mathbf{k}_n) = \epsilon^{n(\Delta-d)}\,\bar{c}_n(\mathbf{k}_1,\ldots,\mathbf{k}_n)$ where $\bar{c}_n$ is finite. Rearranging terms, the inner correlator becomes

$$\rho^{-n(\Delta-d)}\left[\prod_j (1 + F(q_j^2\rho^2))\right]\tilde{c}_n(\rho;\mathbf{k}_1,\ldots,\mathbf{k}_n) = \bar{c}_n(\mathbf{k}_1,\ldots,\mathbf{k}_n)\,. \qquad (10)$$

First consider $(q_j\rho) \ll 1$ for all j. Then the interior correlators at ρ and ρ' are related by a rescaling $(\rho/\rho')^{n(\Delta-d)}$. This is exactly the behavior expected for low energy correlation functions in a Wilsonian effective treatment. We argued in Sec. 1 that the observables on the surface $\partial\mathcal{M}_\rho$ in Poincaré coordinates were smeared at a scale ρ. A Wilsonian treatment requires a rescaling of coordinates to keep the numerical size of the cutoff fixed. Precisely this effect is achieved by the Weyl factor in the metric on $\partial\mathcal{M}_\rho$ which keeps the proper size of the smearing fixed. This in turn results in scaling of the correlators as we flow inwards (to the infrared).

More generally, since $\bar{c}$ on the right-hand side of (10) is independent of ρ, we have an RG equation,

$$\left[\rho\,\frac{\partial}{\partial\rho} - n(\Delta - d)\right]\tilde{c}_n(\rho;\mathbf{k}_1,\ldots,\mathbf{k}_n) + \left[\sum_j \rho\,\frac{\partial}{\partial\rho}\,\ln[1 + F(q_j^2\rho^2)]\right]\tilde{c}_n(\rho;\mathbf{k}_1,\ldots,\mathbf{k}_n) = 0\,. \qquad (11)$$

When all the momenta q are small, the second term vanishes and, as expected, we have the RG equation for pure scaling of infrared Wilsonian correlators. Violations of scaling appear in the second term and are suppressed at low momenta.

Bulk field equations from CFT?: In the semiclassical limit, the interior effective theories that we have constructed are related to the exterior CFT by a renormalization group transformation, suggesting a direct relation between the bulk field equations and the RG equations in the CFT. This is at first surprising since the bulk field equations are second order while the RG equations are first order. However, there is no real conflict because demanding regularity of the bulk solutions in Euclidean space eliminates one solution, making the equations effectively first order. Related observations have been made in [3].

The connection can be made more explicit by recalling the correspondence between boundary behavior of the bulk fields in AdS_{d+1} and sources and operators in the gauge theory [6,7,10–12]. Up to a ρ dependent scaling, sources correspond to the boundary values of bulk fields while operators correspond to their radial derivatives. Schematically, $J \sim \Phi$, $\mathcal{O} \sim \rho\, \partial_\rho \Phi$. In the CFT, J appears as a coupling to the gauge invariant operator $\mathcal{O}$ of the form $\int J(\mathbf{b})\mathcal{O}(\mathbf{b})$. Now consider the structure of the bulk equation for a free scalar field of mass m: $[\rho^2 \partial_\rho^2 + (1 - d)\rho\,\partial_\rho - \vec{k}^2 \rho^2 - m^2]\Phi(\rho)e^{i\vec{k}\cdot\vec{x}} = 0$. If we use the relations above for J and $\mathcal{O}$, we find that the field equation takes the form

$$\left[\rho \frac{\partial}{\partial \rho} + d_0\right]\mathcal{O} - [d_1 + d_2 \vec{k}^2 \rho^2] J = 0 . \qquad (12)$$

Again, we are being schematic—d_0, d_1, and d_2 are constants. The source J is not an independent variable since it determines the expectation value for $\mathcal{O}$. In momentum space, J can be expressed as $\mathcal{O}$ times a function of $\vec{k}^2$. Using this, we find that (12) has the same form as (11) with $n = 1$.

To make this connection precise, various issues such as the scheme dependence of the RG equations must be confronted. Nevertheless, there is reason to hope that the field equations of supergravity can be derived from the CFT via the renormalization group. Work in this direction is in progress.

(3) Discussion: geometry and RG flows.—We have argued that there is a natural way to define an "interior" holographic correspondence between physics inside finite volumes $\mathcal{M}_\rho$ and a theory on the boundary $\partial\mathcal{M}_\rho$. The correlation functions of the interior theory are related to the exterior observables by a coarsening transformation. A given family of foliating surfaces then leads to a particular flow of smeared observables summarized by a renormalization group equation. Changing the foliation leads to a different flow. In fact, we are learning that spacetime geometry arises in a holographic context as the geometry of the space of RG flows.

Consider a CFT defined on a plane and a family of theories derived from it by coarsening transformations. Concretely, let $\phi(\mathbf{b})$ be a field in the CFT, and define coarsened fields $\phi(\rho; \mathbf{b})$ by convolving Φ against a kernel K_ρ which has a characteristic scale ρ. As ρ increases from 0 to ∞, we arrive at a family of smeared theories. In some natural sense there should be a geometry on this "stack" of theories. First of all, a coarsening transformation should be accompanied by a rescaling of lengths, and that is implemented by rescaling the metric of the smeared theories. In addition, we would like a notion of distance or separation between the original CFT and its cousins that depends on the coarsening parameter ρ. For the class of kernels inspired by AdS/CFT, we have learned that there is a natural distance, and it is given by the geodesic length between the fixed ρ Poincaré surfaces. In this sense, anti–de Sitter spaces induce a geometry on a certain class of RG flows of the dual CFTs.

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