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Complete Characterization of Arbitrary Quantum Measurement Processes

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We examine two simple and feasible practical schemes allowing the complete determination of any quantum measuring arrangement. This is illustrated with the example of parity measurement.

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In the most standard picture of quantum mechanics the statistics of every measurement are governed by the projection of the system state on the orthogonal eigenstates of a set of commuting self-adjoint operators representing the measured observables. This implicitly assumes that the measurement is performed on a closed system.

A more complete and realistic picture must encompass the possibility of controllable as well as unpredictable couplings of the observed system with external agents. This is to say that the measurement is performed, in general, on an open system. Among other consequences, this extends the idea of observables beyond self-adjoint operators, introducing generalized measurements described by positive operator measures (POMs).

In particular, this occurs when a standard measurement is preceded by an interaction of the observed system with other degrees of freedom that are in a fixed and known initial state [1]. On the other hand, the process can also involve uncontrollable influences (usually undesired) of outer degrees of freedom. This is frequently the case with couplings with reservoirs and other mechanisms leading to losses and decoherence effects, for instance. In many practical situations it is not possible to predict which external variables are involved or the way they affect the performance of the measurement. In other words, to some extent, the real measurement differs from the intended one in an unpredictable way.

In this paper we present two simple and feasible practical procedures that allow us to determine completely any quantum measurement process. The objective of such a characterization is to obtain in practice the actual POM governing the statistics. This would allow one to ascertain to what extent the planned performance is reached by revealing undesired deviations.

The system, which is the object of the observation, and the external variables involved are described by the Hilbert spaces $\mathcal{H}_s$ and $\mathcal{H}_a$, respectively. Since the total Hilbert space $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_a$ represents by definition a closed system, the system-environment interaction can always be implemented by a unitary operator U acting on $\mathcal{H}$. The system is initially in an arbitrary state with density matrix ρ_s, while the external variables will be in some state ρ_a that does not depend on ρ_s. After the interaction, some set of compatible observables are measured on the output state $U\rho_s\rho_a U^\dagger$. The statistics of the measurement are given by the projection on a set of orthonormal vectors $|k\rangle$ whose span need not coincide with $\mathcal{H}_s$:

$$\mathcal{P}(k) = \mathrm{tr}_{s,a}(|k\rangle\langle k|U\rho_s\rho_a U^\dagger)\,, \tag{1}$$

where $\mathcal{P}(k)$ is the probability of the outcome k. This is a standard ideal measurement because $\mathcal{H}_s \otimes \mathcal{H}_a$ is a closed system. Since ρ_a is fixed and does not depend on ρ_s, the information provided by the measurement can be regarded as information about ρ_s. This interpretation can be expressed explicitly by rearranging Eq. (1) in the form

$$\mathcal{P}(k) = \mathrm{tr}_s[\Delta(k)\rho_s]\,, \tag{2}$$

where

$$\Delta(k) = \mathrm{tr}_a(U^\dagger|k\rangle\langle k|U\rho_a) \tag{3}$$

is a POM acting on $\mathcal{H}_s$. These expressions represent a standard measurement when the system is decoupled from the environment or its effect is trivial. We have a generalized measurement whenever U is nontrivial, provided that ρ_a and U are completely known in advance [1]. This formulation comprises any uncontrollable source of uncertainty or error that might be present in a real measurement if we take into account that ρ_a and/or U can be partially or completely unknown.

Such an indetermination can be removed by the practical characterization of the process. The purpose is to find procedures that enable the knowledge of the actual $\Delta(k)$ without requiring any prior knowledge about ρ_a and U. Equivalently, the measurement is completely characterized once we are able to predict successfully the statistics $\mathcal{P}(k)$ for every ρ_s.

For the sake of simplicity, in what follows we shall consider that $\mathcal{H}_s$ and $|k\rangle$ are associated with an unbounded continuous degree of freedom describable by adimensional Cartesian variables q and p with commutation relation $[q, p] = i$. It will be seen that the removal of these conditions is straightforward. Among other examples, this includes the one-dimensional motion of a trapped ion, where q and p are position and linear momentum, and a single mode of the electromagnetic field, where q and p are field quadratures.

The first characterization procedure we will examine relies on the controlled variation of the initial state in a suitable domain (Fig. 1). A very simple and feasible choice is the coherent states $\rho_s = |\alpha\rangle\langle\alpha|$ defined by the eigenvalue equation $a|\alpha\rangle = \alpha|\alpha\rangle$, where a is the annihilation or ladder operator $a = \lambda q + ip/(2\lambda)$, with λ a suitable constant [2–4]. Since the complex parameter α is allowed to vary, the statistics of the measurement $Q(k, \alpha)$ depend on the variables k and α, and is given by

$$Q(k, \alpha) = \langle\alpha|\Delta(k)|\alpha\rangle. \tag{4}$$

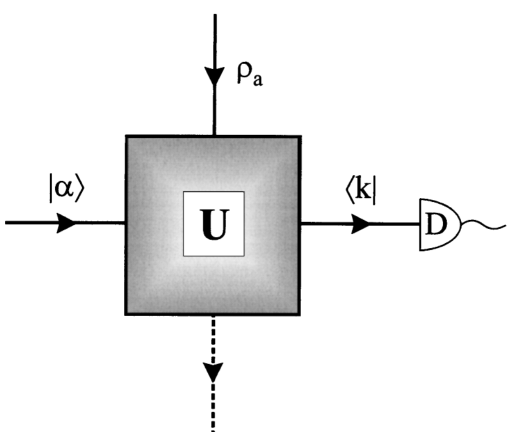

FIG. 1. Scheme for the practical determination of a measurement process. An input coherent state $|\alpha\rangle$ is coupled with auxiliary degrees of freedom in the state ρ_a. After the interaction, a standard measurement is performed in some of the degrees of freedom. The statistics are given by projection on the orthogonal vectors $|k\rangle$. The dashed line represents the degrees of freedom which are not measured.

We can see that for fixed outcome k this is the Q function of $\Delta(k)$, which is an informationally complete representation on phase space of any operator [2–4]. Then, the probabilities $Q(k, \alpha)$ determine completely $\Delta(k)$ and, therefore, the measurement process.

As mentioned above, this procedure can be easily adapted to other circumstances. We can briefly examine the case when $\mathcal{H}_s$ is finite dimensional, as it occurs for spins or for the internal state of atoms when a finite set of energy levels is involved. In such a case the previous procedure is still valid if we replace $|\alpha\rangle$ by SU(2) coherent states

$$|\zeta\rangle = \frac{1}{(1 + |\zeta|^2)^j} \sum_{m=-j}^{j} \binom{2j}{j + m}^{1/2} \zeta^{j+m}|j, m\rangle, \tag{5}$$

where ζ is a complex parameter and the $2j + 1$ vectors $|j, m\rangle$ $(m = -j, -j + 1, \ldots, j)$, for integer or half-integer j, are an orthonormal basis of the corresponding Hilbert space [2].

In the preceding scheme the complex amplitude α (or ζ) of the input coherent state must be varied and its value has to be monitored. Next, we examine a second scheme, where the input state need not be varied. To this end, let us consider the following state $|\xi\rangle$ in the Hilbert space $\mathcal{H}_s \otimes \mathcal{H}_i$:

$$|\xi\rangle = \sqrt{1 - |\xi|^2} \sum_{n=0}^{\infty} \xi^n |n, n\rangle, \tag{6}$$

where $\mathcal{H}_i$ represents an auxiliary degree of freedom, ξ is a complex parameter with $|\xi| < 1$, and $|n, m\rangle = |n\rangle_i \otimes |m\rangle_s$ is the number basis. The space $\mathcal{H}_s \otimes \mathcal{H}_i$ can represent the movement of a trapped ion along two orthogonal directions or two modes of the electromagnetic field. In the case of trapped ions, $|n, m\rangle$ are energy eigenstates of a two-dimensional harmonic trap and the state (6) can be prepared as shown, for example, in Ref. [5]. In the case of the electromagnetic field, the states $|n, m\rangle$ are photon-number states and $|\xi\rangle$ is generated in spontaneous parametric down-conversion in a nonlinear crystal [4].

Together with the measurement $\Delta(k)$ on $\mathcal{H}_s$, we consider a simultaneous measurement performed on $\mathcal{H}_i$, as shown in Fig. 2. We assume that this additional measurement allows us to reconstruct the state on this degree of freedom, as can be achieved by different standard practical arrangements. For definiteness, we assume that the statistics of the measurement are proportional to the projection on a coherent state $|\alpha\rangle \in \mathcal{H}_i$, where the complex parameter α represents the outcomes. This measurement can be implemented for field modes [6] as well as for trapped ions [7].

The joint statistics associated with the two simultaneous measurements are

$$\begin{aligned} \mathcal{P}(k, \alpha) &= \mathrm{tr}_{s,i}[|\alpha\rangle\langle\alpha|\Delta(k)|\xi\rangle\langle\xi|] \\ &= \mathrm{tr}_s[\Delta(k)\langle\alpha|\xi\rangle\langle\xi|\alpha\rangle], \end{aligned} \tag{7}$$

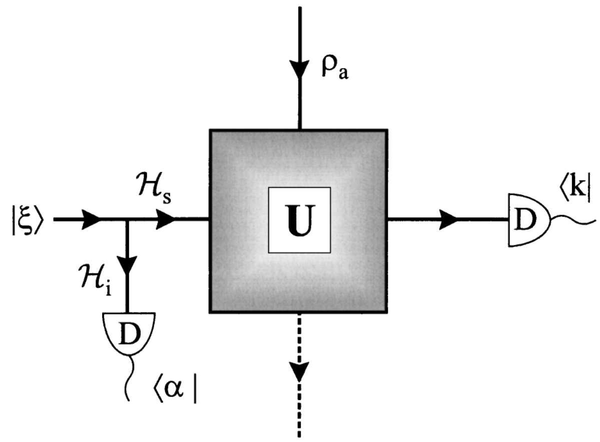

FIG. 2. Arrangement to determine a measurement process without varying the initial state $|\xi\rangle \in \mathcal{H}_s \otimes \mathcal{H}_i$. To this end, an additional measurement is performed in the degrees of freedom represented by $\mathcal{H}_i$. The statistics of such a measurement are given by the projection on the coherent states $|\alpha\rangle \in \mathcal{H}_i$.

where the last equality follows because $\Delta(k)$ does not act on $\mathcal{H}_i$. Since $|\xi\rangle$ is a vector in $\mathcal{H}_i \otimes \mathcal{H}_s$ and $|\alpha\rangle$ is a vector in $\mathcal{H}_i$, the projection $\langle\alpha|\xi\rangle$ in Eq. (7) gives a vector in $\mathcal{H}_s$,

$$\langle\alpha\,|\,\xi\rangle = \sqrt{1 - |\xi|^2}\, e^{-|\alpha|^2(1-|\xi|^2)/2}|\xi\alpha^*\rangle, \qquad (8)$$

where $|\xi\alpha^*\rangle$ is a coherent state in $\mathcal{H}_s$. We then have

$$\mathcal{P}(k,\alpha) = (1 - |\xi|^2)e^{-|\alpha|^2(1-|\xi|^2)}\langle\xi\alpha^*|\Delta(k)|\xi\alpha^*\rangle, \ (9)$$

or, equivalently,

$$Q(k,\alpha) = \frac{1}{1 - |\xi|^2}\, e^{|\alpha|^2(1-|\xi|^2)/|\xi|^2}\,\mathcal{P}(k,\alpha^*/\xi^*). \ \ (10)$$

Therefore, this scheme also provides the complete characterization of the process via the same function $Q(k,\alpha)$.

This arrangement works because of the strong correlations between the $\mathcal{H}_s$ and $\mathcal{H}_i$ variables in the state $|\xi\rangle$. This correlation, which is clear in the $|n,m\rangle$ basis, extends as well to any other basis. Because of this, the realization of a measurement on $\mathcal{H}_s$ or $\mathcal{H}_i$ reduces the state in the other Hilbert space to a state strongly related to the measurement performed and its outcome [8]. Since we are dealing with two simultaneous measurements, we can express the result obtained in two different but equivalent ways.

If we focus on a particular outcome k of the measurement performed on $\mathcal{H}_s$, we find that the reduced state in $\mathcal{H}_i$ is proportional to $\xi^{a^\dagger a}\Delta^*(k)\xi^{*a^\dagger a}$, where $\Delta^*(k)$ denotes the complex conjugate of $\Delta(k)$ in the number basis [8]. The measurement in $\mathcal{H}_i$ then allows us to reconstruct $\Delta(k)$ via its Q function.

On the other hand, if we consider a particular outcome α for the measurement in $\mathcal{H}_i$, the reduced state in $\mathcal{H}_s$ is a coherent state, as Eq. (8) shows. Each outcome means a different coherent state incident at the input of the measurement process, so this arrangement works as the first one analyzed above (Fig. 1). However, we stress that in this last case the incident state on the arrangement is always the same and nothing needs to be varied. The outcome of the measurement in $\mathcal{H}_i$ monitors automatically the actual input coherent state in $\mathcal{H}_s$.

Finally, it can be worth examining the practical performance of this scheme. Two sources of uncertainty will affect the accuracy of the characterization. These are the sampling error caused by the finite number of scanned values of α and the statistical fluctuations caused by the limited set of repeated measurements at each α point. This will limit the knowledge of the function $Q(k,\alpha)$, which in turns implies some amount of uncertainty when predicting the results of potential measurements on arbitrary input states. We must stress that the influence of any other agents usually regarded as sources of error, such as low detection efficiencies, for instance, does not imply any imperfection or uncertainty in our case, since they are part of the actual measuring arrangement being characterized.

This can be illustrated by means of a particular example. We will assume that the apparatus measures parity. In such a case, there are only two possible outcomes: $k = +,-$, with $\Delta(+) = \sum_{n=0}^{\infty}|2n\rangle\langle 2n|$ and $\Delta(-) = I - \Delta(+)$. We can also consider that during the measurement the system is superposed with thermal noise so that the actual positive operator measure is no longer given by projection on pure states but on the accordingly thermalized states. Then, the exact Q function to be determined is

$$Q(+,\alpha) = \frac{1}{2}\left(1 + \frac{1}{1 + 2\bar{n}}\, e^{-2|\alpha|^2/(1+2\bar{n})}\right), \qquad (11)$$

where $\bar{n}$ is the average number of thermal photons added [9].

In Fig. 3 we show the typical result of a finite sampling of $Q(+,\alpha)$ affected by statistical fluctuations. In order to infer the actual $\Delta(+)$ from the registered data, different algorithms may be used [10–12]. According to the procedure followed in this paper, we could extract an analytic function $\tilde{Q}(\alpha)$, estimating the true but unknown $Q(+,\alpha)$.

After a set of outcomes such as the ones represented in Fig. 3, it can be inferred that the reconstructed $\tilde{Q}$ function that will fit best to the experimental data will be of the form $\tilde{Q}(\alpha) = c_0 + c_1\exp(-c_2|\alpha|^2)$, where the parameters c_0, c_1, and c_2 have to be estimated. This fitting should always be compatible with quantum mechanics [12]. This means that $\tilde{Q}$ must represent probabilities and, being a Q function, it cannot be arbitrarily narrow. Then, the parameters c_0, c_1, and c_2 cannot take arbitrary values, and suitable constraints should be taken into account in such a fitting.

Concerning errors, meaningful conclusions can be obtained using the method of least squares [10]. Although

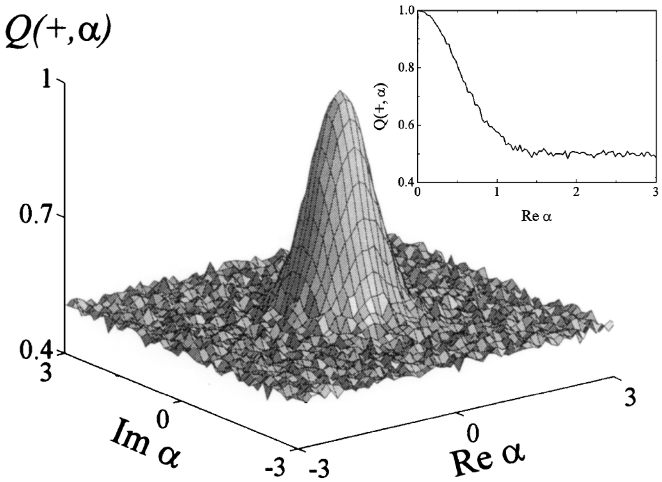

FIG. 3. Plot of the $Q(+,\alpha)$ function affected by statistical fluctuations and evaluated on a square grid of 9×10^2 equally spaced α points. The inset shows a radial section with 10^2 points. The statistical fluctuations correspond to 3×10^3 measurements for each α.

this method can lead to $\tilde{Q}$ functions that do not satisfy the mentioned constraints, it can still serve to suitably estimate the accuracy of the reconstruction and the uncertainty δ on the prediction of future measurement results $\mathrm{tr}[\rho_s \Delta(+)]$. For example, it can be easily seen that there is an upper bound for δ^2 that is proportional to $\mathrm{tr}[\rho_s (a^\dagger a)^2]$ and inversely proportional to the total number of measurements performed.

In conclusion, let us regard the results of this work from another perspective. The standard idea of measurement has been enlarged since it has been shown that suitable arrangements enable the determination of the quantum state the system is in (provided it can be repeatedly prepared) [13]. This goes beyond the standard simultaneous determination of compatible observables because it gives the statistics of all observables at once. Very recently it has been further extended by showing that there are practical schemes allowing the experimental determination of input-output transformations [14]. In this context, the two schemes studied above can be interpreted as a further extension of this concept to include measuring arrangements. In fact, it appears that it is much simpler to measure a POM than a quantum state.

The measurement of transformation operators demonstrated in Ref. [14] might also serve to deduce the effective POM, provided that the actual vectors $|k\rangle$ determining the final measurement were known. In comparison with this possibility, the method presented in this paper does not require any previous knowledge about $|k\rangle$.

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