Sunspot Cycle: A Driven Nonlinear Oscillator?

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A property of nonlinear oscillators—mutual dependence between their instantaneous amplitude and frequency—is tested in the yearly and monthly records of the sunspot numbers using the histogram-adjusted isospectral surrogate data and the Barnes model as the autoregressive moving average surrogates. The instantaneous amplitudes and frequencies are obtained by means of the analytic signal approach using the discrete Hilbert transform. In several tests the amplitude-frequency correlation has been found significant on levels ranging from $p < 0.03$ to $p < 0.07$, which supports the hypothesis of a driven nonlinear oscillator as a mechanism underlying the sunspot cycle.

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The historical data of the sunspot index have been attracting researchers for more than a century. In 1852 Wolf [1] reported the now well-known 11-year cycle. Of course, the sunspot cycle is not strictly periodic, but fluctuations in its amplitude as well as in its frequency (i.e., in the cycle duration) occur. Therefore researchers have turned towards stochastic models in order to make predictions of the future behavior of the sunspot cycle (see [2], and references therein). On the other hand, development in nonlinear dynamics and theory of deterministic chaos, namely methods and algorithms for analysis and prediction of (potentially) nonlinear and chaotic time series, have naturally found their way into the analyses of the sunspot series. Several authors ([3,4], and references therein) have claimed an evidence for the deterministic chaotic origin of the sunspot cycle, based on estimations of correlation dimension, Lyapunov exponents, and an increase of a prediction error with a prediction horizon. The dimensional algorithms, however, have been found unreliable when applied to relatively short experimental data, and properties consistent with stochastic processes (colored noises) such as autocorrelations can lead to spurious convergence of dimensional estimates [5]. Similar behavior has been observed also for Lyapunov exponent estimators [6,7]. And the increase of a prediction error with an increasing prediction horizon is not a property exclusive for chaos, but it can also be observed in systems with a deterministic skeleton and an intrinsic stochastic component ("dynamical noise").

Looking for deterministic chaos in experimental time series, a statistical technique of surrogate data [8] based on rejection by a statistical test of an appropriate null hypothesis has become a standard in nonlinear time series analysis. Applying this approach, the deterministic chaotic origin of the sunspot cycle has not been confirmed, or, at least, the authors are not aware of any published study presenting a solid statistical evidence for chaos in the sunspot cycle. Some researchers [9,10] have been able to detect (unspecified) nonlinearity in this data, however, no specific properties supporting the hypothesis of a low-dimensional chaotic attractor have been found. It is also questionable whether the hypothesis of a closed stationary autonomous system possessing a strange attractor is a reasonable explanation of the dynamics underlying the solar cycle. It might be reasonable, on the other hand, to search for a weaker hypothesis than a chaotic attractor, which, however, would provide a physical meaning to the previously confirmed (unspecified) nonlinearity [9,10] in the sunspot cycle dynamics. In particular, we will demonstrate that the sunspot cycle possesses a significant correlation between its instantaneous amplitude and frequency, which is a property of nonlinear oscillators, and thus we will provide an evidence for a nonlinear oscillator (with possibly random driving) underlying the dynamics of the sunspot cycle, unless the amplitude-frequency relation is explained by a different mechanism.

The instantaneous amplitude and phase of a signal $s(t)$ can be determined by using the analytic signal concept of Gabor [11], recently introduced into the field of nonlinear dynamics within the context of chaotic synchronization by Rosenblum *et al.* [12]. The analytic signal $\psi(t)$ is a complex function of time defined as

$$\psi(t) = s(t) + j\hat{s}(t) = A(t)e^{j\phi(t)}, \tag{1}$$

where the function $\hat{s}(t)$ is the Hilbert transform of $s(t)$

$$\hat{s}(t) = \frac{1}{\pi}\,\mathrm{P.V.}\int_{-\infty}^{\infty}\frac{s(\tau)}{t-\tau}\,d\tau\,. \tag{2}$$

(P.V. means that the integral is taken in the sense of the Cauchy principal value.) $A(t)$ is the instantaneous

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amplitude and the instantaneous phase $\phi(t)$ of the signal $s(t)$ is

$$\phi(t) = \arctan \frac{\hat{s}(t)}{s(t)}. \tag{3}$$

The instantaneous frequency $\omega(t)$ is the derivative $\dot{\phi}(t)$ of the phase $\phi(t)$.

As a demonstrative example of a nonlinear oscillator (not a model for the sunspot cycle) we will consider the Duffing oscillator

$$\ddot{x} + 0.05\dot{x} + x + x^3 = F(t). \tag{4}$$

If $F(t) = 0$ and without the cubic member x^3, Eq. (4) represents a damped linear oscillator with a constant frequency and an exponentially decreasing amplitude. The presence of the nonlinear (cubic) member x^3 in Eq. (4) leads to a time dependent frequency, and considering again $F(t) = 0$, both the amplitude $A(t)$ and frequency $\omega(t)$ exponentially decrease (Figs. 1a and 1b). Now, consider that the nonlinear oscillator (4) is driven by a random driving force $F(t)$. The relation between $A(t)$ and $\omega(t)$ is a nonlinear function and may vary in time, however, the level of the correlation between $A(t)$ and $\omega(t)$ depends on the driving force: With a relatively weak driving (Fig. 1c), $A(t)$ and $\omega(t)$ are almost perfectly correlated (Fig. 1d), with a stronger driving force $F(t)$ (Fig. 1e) some differences between $A(t)$ and $\omega(t)$ emerge, however, $A(t)$ and $\omega(t)$ are still correlated (Fig. 1f).

A possible amplitude-frequency correlation (AFC) in the sunspot cycle, in particular, the importance of the amplitude in determining the length of the related cycle has already been noted in the 1930s by Waldmeier [13] and recently discussed by Hathaway *et al.* [14]. In this Letter we demonstrate that the amplitude-frequency correlation found in the sunspot cycle is probably a nonrandom phenomenon and propose its explanation by an underlying nonlinear dynamical system. We have used the yearly and monthly sunspot numbers from the Sunspot Index Data Center [15].

The yearly sunspot numbers series from the years 1700–1997 A.D. (Fig. 2a) has been filtered by a simple moving average (MA) bandpass filter: First, the MA's from a 13-sample window have been subtracted from the data in order to remove slow processes and trends, and then a 3-sample MA smoothing has been used in order to remove high-frequency components and noise. Then the discrete version of the Hilbert transform (2) using the window length of 25 samples has been applied in order to obtain the instantaneous amplitude $A(t)$ and the instantaneous phase $\phi(t)$. For obtaining a more robust estimation of the instantaneous frequency $\omega(t)$ than the one yielded by a simple differencing of the phase $\phi(t)$, the robust linear regression [16] in a 7-sample moving window has been used. Finally, the series of $A(t)$ and $\omega(t)$ have been smoothed using a 13-sample MA window. The resulting series of the instantaneous amplitude and frequency of the yearly sunspot numbers, plotted in Fig. 2b, yield the crosscorrelation equal to 0.505. Does this value mean that the amplitude and frequency of

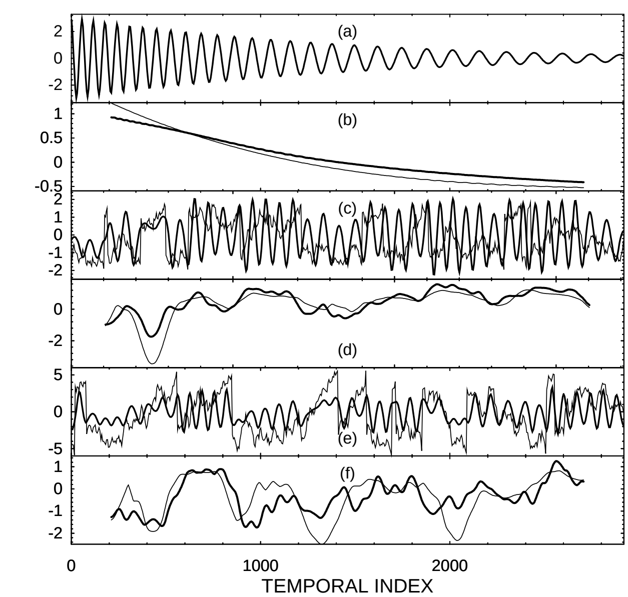

FIG. 1. (a) A solution of the nonlinear Duffing oscillator without any external driving force, and (b) the related instantaneous amplitude (thick line) and frequency (thin line). (c) A solution (thick line) of the nonlinear Duffing oscillator with a random driving force $F(t)$ (thin line), and (d) the related instantaneous amplitude (thick line) and frequency (thin line). (e) A solution (thick line) of the nonlinear Duffing oscillator with a stronger random driving force $F(t)$ (thin line), and (f) the related instantaneous amplitude (thick line) and frequency (thin line).

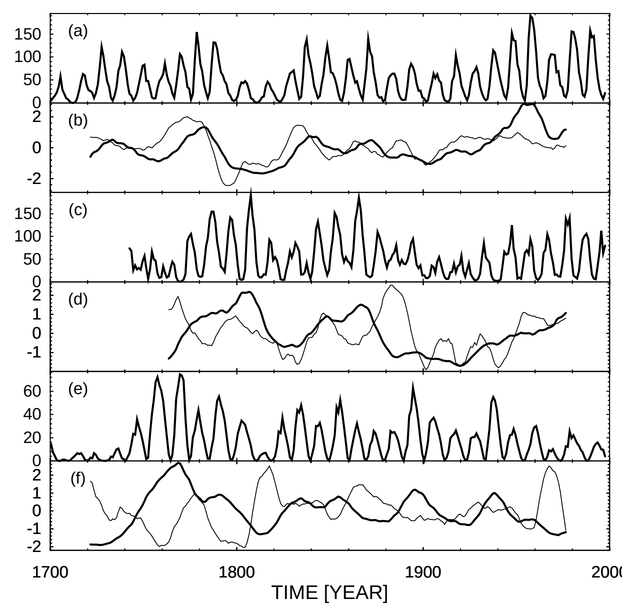

FIG. 2. (a) The yearly sunspot numbers series (1700–1997 A.D.) and (b) the related instantaneous amplitude (thick line) and frequency (thin line). (c) A realization of the HAFT surrogate data for the "last" 256 samples, and (d) the related instantaneous amplitude (thick line) and frequency (thin line). (e) A 298-sample realization of the Barnes model and (f) the related instantaneous amplitude (thick line) and frequency (thin line).

the sunspot cycle are correlated as a consequence of an underlying dynamics, or could this correlation occur by chance? Searching for an answer, we test the statistical significance of this correlation by using the approach of surrogate data [8]. We generate a large number of realizations of processes which mimic some properties of the sunspot numbers series, however, which do not possess any systematic (nonzero) AFC. A nonzero amplitude-frequency correlation in such processes can only occur randomly in some of their realizations. Thus we estimate the probability of a random occurrence of the AFC found in the sunspot data, considering the chosen null hypothesis (surrogate model).

In the first kind of the surrogate tests we apply isospectral, or Fourier transform (FT) surrogates and histogram adjusted isospectral (HAFT) surrogates. (In [8] the term "amplitude-adjusted"—AAFT—surrogates is used.) The isospectral surrogates are realizations of a linear stochastic process which has the same spectrum as the studied data. In this case, using the fast Fourier transform (FFT) [16] which requires the number of samples equal to a power of 2, we perform two tests, using the "first" and the "last" 256 samples, i.e., the subseries of the whole 298 sample series obtained by cutting away 42 samples at the end, or at the beginning, respectively, from the whole yearly sunspot numbers record. Thus, in each test, the surrogate data replicate the sample spectrum of the related 256-sample subseries. The FT surrogates are obtained by computing FFT of the raw data, then randomizing the phases of the Fourier coefficients, but keeping the magnitudes (of the Fourier coefficients, i.e., the spectrum) unchanged, and computing the inverse FFT into the time domain. The resulting series is a realization of a linear stochastic process with the same spectrum as the sample spectrum of the related segment of the sunspot numbers series. In other words, the FT surrogates are data with cycles oscillating with the same frequencies as the sunspot cycles, however, not possessing any systematic amplitude-frequency correlation. Using different sets of the random Fourier-coefficient phases, different realizations of the surrogate data can be generated.

The FT surrogates tend to have a Gaussian distribution which is not always the case of the tested data. In order to avoid a possible influence of different histograms of the data and the surrogates, the histogram adjusted FT surrogates are constructed. In this case, the raw data undergo a nonlinear transformation which leads to a Gaussian distribution of the transformed data ("gaussianization"—see [9] and references within). The gaussianized data are used to generate the FT surrogates as described above, and the obtained surrogate data are transformed in order to have the same histogram as the original raw data.

The (HA)FT surrogates are generated from the raw (unfiltered) 256-sample segments of the sunspot data. Also, the 256-sample subseries are used for estimating the amplitude-frequency correlation related to the particular subseries, applying the procedures described above. Then, each realization of the (HA)FT surrogates, generated with respect to the raw data, undergoes the same processing as the raw data, i.e., the MA bandpass filtering, the Hilbert transform and the robust linear regression for the $\omega(t)$ estimation, and the final $A(t)$ and $\omega(t)$ smoothing are performed before computing the AFC for each surrogate realization. Then the *absolute* values of the AFC's for 150 000 surrogate realizations are evaluated in order to assess the significance of the related AFC value found in the sunspot data. The first 256-sample subseries of the sunspot yearly numbers yields the AFC equal to 0.605, while the mean value of the *absolute* AFC for the HAFT surrogate set is 0.26 with the standard deviation (SD) equal to 0.17. In usual surrogate tests the significance is derived from the difference between the data value and the surrogate mean, divided by the surrogate SD, provided normal distribution of the surrogate values. Having generated the large amount of the surrogate replications, here we directly estimate the p value of the test, i.e., the probability that the assessed correlation occurred by chance (randomly) within the chosen null hypothesis (surrogate model), by simply counting the occurrences in the surrogate set of absolute AFC values greater than or equal to the assessed raw data value, i.e., 0.605 in this case. The number obtained is 3637, which is equal to 2.43%. Statistically speaking, the test result is significant on $p < 0.03$, or, in other words, the probability that the amplitude-frequency correlation found in the studied segment of the sunspot data occurred by chance (as a random event) is smaller than 3%.

Processing the "last" 256-sample segment of the yearly sunspot numbers, the obtained AFC is equal to 0.532, while the values from the HAFT surrogates are the same as above, however, the p value in this case is 6.58%. Still, we can conclude that the test result is significant on $p < 0.07$. An example of the HAFT surrogate realization is plotted in Fig. 2c, its instantaneous amplitude and frequency in Fig. 2d.

The results from the tests using simple FT surrogates (i.e., without the histogram adjustment) are practically the same as those from the above HAFT surrogates. Testing the monthly sunspot numbers (1749–1997 A.D.), the segments of ("first" and "last") 2048 samples were used. The same data processing has been applied as described above in the case of the yearly data with the window lengths equivalent in real time, i.e., multiplied by 12 in number of samples. The obtained results are perfectly equivalent to those yielded by the yearly data, i.e., $p < 0.03$ and $p < 0.07$ for the first and the last 2048-sample segments, respectively.

The HAFT surrogates replicate the spectrum and the histogram of the sunspot data, however, do not reflect temporal asymmetry typical for the sunspot cycle behavior. Therefore, we employ also another kind of surrogate data, generated by the Barnes model [17] which incorporates the structure of an autoregressive moving average

ARMA(2,2) model with a nonlinear transformation:

$$z_i = \alpha_1 z_{i-1} + \alpha_2 z_{i-2} + a_i - \beta_1 a_{i-1} - \beta_2 a_{i-2}, \quad (5)$$

$$s_i = z_i^2 + \gamma(z_i^2 - z_{i-1}^2)^2, \quad (6)$$

where $\alpha_1 = 1.906\,93$, $\alpha_2 = -0.987\,51$, $\beta_1 = 0.785\,12$, $\beta_2 = -0.406\,62$, $\gamma = 0.03$ and a_i are IID Gaussian random variables with zero mean, and SD $= 0.4$. The nonlinear transformation (6) ensures that the generated series remains asymmetric and positive and tends to increase more rapidly than it decreases (Fig. 2e), which are the properties observed in the sunspot data. Moreover, the stochastic Barnes model can mimic the correlation integrals [18] and the phase portraits [19] obtained from the sunspot series. No systematic amplitude-frequency correlation, however, is present in the series obtained from the Barnes model (Fig. 2f). In the test, 150 000 298-sample realizations of the Barnes model have been generated and processed by the same way as the sunspot series. The mean absolute AFC is equal to 0.21, SD $=$ 0.15, comparison with the AFC obtained for the whole 298-sample yearly sunspot series (AFC $= 0.505$) yields the p value equal to 4.36%. Thus, considering the Barnes model, the probability that the whole yearly sunspot series AFC $= 0.505$ occurred by chance is $p < 0.05$.

Using two different types of stochastic models which replicate some properties of the sunspot cycle we have obtained a statistical support for the hypothesis that the amplitude-frequency correlation observed in the sunspot cycle did not occur by chance (as a random event) but is probably a property of an underlying dynamical mechanism. Well-known systems, possessing this property, are nonlinear oscillators, in which a significant AFC can be observed also in cases of external, even random, driving force. Therefore the presented results can be considered as a statistical evidence for a nonlinear oscillator (with possibly random driving) underlying the dynamics of the sunspot cycle, unless the amplitude-frequency relation is explained by a different mechanism.

Although no particular model for the solar cycle has been proposed here, the presented rigorous statistical evidence for a nonlinear dynamical mechanism underlying the sunspot cycle can be understood as a first step in bridging the gap between statistical analyses of the experimental sunspot data (dominated by linear stochastic methods) and physical models such as nonlinear dynamo models [20,21] (developed only on a qualitative level). Statistical comparison of the data with the latter models would follow, with the aim to construct a realistic data-driven model for the solar cycle.

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