Keyhole Look at Lévy Flights in Subrecoil Laser Cooling

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We propose a method to measure the waiting-time distribution of trapped atoms in subrecoil laser cooling.

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Lévy flights [1,2] manifest themselves in various phenomena ranging from albatrosses searching for food, fluctuations in share prices, and to turbulence. Most recently, they have entered the field of quantum optics through laser cooling [3], the quantum kicked rotor [4], and the motion of atoms in optical lattices [5]. Impressive laser experiments [6,7] have demonstrated Lévy flights. In contrast to the anomalous transport in optical lattices the stochastic variable obeying the Lévy law in the realm of subrecoil laser cooling [8] based on velocity selective coherent population trapping (VSCPT) is of a different nature. It is the waiting time [3] for particles trapped in the vicinity of zero momentum. The waiting-time distribution $P(t) \propto t^{-3/2}$ shows a slowly decreasing power-law tail [3]. This distribution belongs [2] to the attraction basin of the Lévy law $L_{\mu,1}$ with $\mu = 1/2$. In the present paper we suggest a simple method to observe directly the Lévy statistics originating in VSCPT laser cooling.

We consider an additional decay channel and identify the following features: (i) The extra decay acts like a velocity filter, (ii) the counting rate in the decay channel describes asymptotically the waiting-time distribution for the trapping state, and (iii) this distribution satisfies the same Lévy statistics $L_{1/2,1}$ as the closed three-level system [3]. Hence the decay channel is the keyhole which allows us to see Lévy flights.

The standard scheme of 1D subrecoil laser cooling [8] relies on the interaction of a three-level Λ atom with two counterpropagating light beams. For the sake of simplicity we assume that they have equal intensities and are resonant with the atomic transitions between the two sublevels $|g_+\rangle$ and $|g_-\rangle$ of the ground state and the excited state $|e_0\rangle$. The latter decays to the ground state with the rate γ.

Our ultimate goal is to measure the waiting-time distribution for trapped particles. For this purpose we consider an open Λ system [9] where the excited state has an additional decay channel [10] to a nondetected level $|nd\rangle$ with the rate $\gamma_0 \ll \gamma$. Hence the total width of the excited state is $\Gamma \equiv \gamma + \gamma_0$. We do not address the physical mechanism responsible for the decay channel to the outside of the Λ system. We need a keyhole but its details do not matter.

The behavior of the system follows from the quantum equations for the atomic density matrix taking into account the recoil effect due to induced and spontaneous transitions [11]. For a detailed derivation of these equations for the closed Λ system we refer to Ref. [12].

It is convenient to use the state basis $|1\rangle \equiv |g_-, p - \hbar k\rangle$, $|2\rangle \equiv |g_+, p + \hbar k\rangle$, and $|3\rangle \equiv |e_0, p\rangle$ with atomic momenta $p \pm \hbar k$ and p along the laser beams. These states form a closed family with respect to induced transitions in the external field. Spontaneous transitions into the ground states lead due to the random recoil along the laser beams to an incoherent mixing of the families within the range of momenta $|p| \leq \hbar k$. For the sake of simplicity we assume the spontaneous emission to be isotropic. The pumping rate

$$\Gamma_g(p,t) = \frac{\gamma}{4} \int_{-1}^{1} dx\, \rho_{33}(p + \hbar k x, t) \tag{1}$$

of the ground state sublevel with the momentum p is proportional to the partial width γ of the upper level with the population ρ_{33}. In contrast, the damping of the matrix elements ρ_{33}, ρ_{31}, and ρ_{32} is determined by the total width Γ. This is the only difference in comparison with the closed Λ system [13–15].

Throughout the paper we consider the weak saturation regime, when the saturation parameter $g \equiv dE/(\hbar\Gamma)$ with the dipole moment d and the electric field E is small; that is $g \ll 1$. This allows us, provided $t \gg 1/\Gamma$, to eliminate the excited state and to obtain the closed set of equations for the ground state density matrix elements $n(p,t) \equiv \rho_{11}(p,t) + \rho_{22}(p,t)$, $\mu(p,t) \equiv \rho_{21}(p,t) + \rho_{12}(p,t)$ and $i\nu(p,t) \equiv \rho_{21}(p,t) - \rho_{12}(p,t)$. These equations read

$$\frac{\partial}{\partial t} n(p,t) = \Gamma g^2 \left[-F(p,t) + \frac{\gamma}{2\Gamma} \int_{-1}^{1} dx\, F(p + 2\hbar k x, t) \right] \tag{2}$$

with

$$\frac{\partial \mu}{\partial t} + \Gamma g^2 F - \frac{2kp}{m} \nu = 0, \tag{3}$$

and

$$\frac{\partial \nu}{\partial t} + \Gamma g^2 G + \frac{2kp}{m} \mu = 0, \tag{4}$$

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where we have introduced the notation

$$\frac{n + \mu + i\nu}{1 + i\frac{2kp}{m\Gamma}} \equiv F + iG . \tag{5}$$

We note that $\frac{1}{2}(n + \mu) \equiv n_{\rm C}$ and $\frac{1}{2}(n - \mu) \equiv n_{\rm NC}$ are the populations of the coupled state $|\psi_{\rm C}\rangle \equiv (|1\rangle + |2\rangle)/\sqrt{2}$ and noncoupled state $|\psi_{\rm NC}\rangle \equiv (|1\rangle - |2\rangle)/\sqrt{2}$, respectively, and $\mu = n_{\rm C} - n_{\rm NC}$ is the population difference.

In the following analysis we eliminate the variables μ and ν and derive a relation connecting the functions n and F. For this we focus on the time regime $t \sim \tau \equiv 1/(\gamma_0 g^2) \gg 1/(\Gamma g^2)$. These conditions guarantee that (i) the relaxation processes inside the Λ system are much faster than the decay process to the outside of the system, and (ii) enough spontaneous transitions into the ground state occur to redistribute the atoms in momentum space. Under these conditions pumping into the trapping state takes place before the Λ system decays to the level $|nd\rangle$.

For atoms with momenta $p \sim m\gamma/k$ the loss rate to the outside of the Λ configuration is of the order of $\gamma_0\, g^2$. However, in the vicinity of $p = 0$ the loss rate is much smaller and proportional to $p^2\gamma_0 g^2$ [3,12]. It vanishes for the dark state $|\psi_{\rm NC}(p = 0)\rangle$, because the level $|e_0, p = 0\rangle$ is not populated at all and no spontaneous transitions to the level $|nd\rangle$ can occur. Therefore, the decay channel shows velocity selectivity and acts like a velocity filter. On the time scale $t \geq \tau$, the decay channel influences of course the kinetics, but it does not change the stochastic properties of the waiting time for trapped particles as we show below.

Moreover, we now consider Eqs. (2)–(5) in a regime where the Doppler shift does not play an essential role. This is the domain $t \ll \gamma/(g\omega_R)^2$ where $\omega_R \equiv \hbar k^2/(2m)$ denotes the recoil frequency. Indeed we recall from Eq. (2) that the characteristic momentum scale $\delta p_{\rm dif}(t)$ for the function $F(p,t)$ is given by the usual diffusion relation [12,16] $\delta p_{\rm dif}(t) \sim \hbar k\sqrt{\gamma g^2 t}$. This provides [3,14,17] the estimate $2kp/(m\Gamma) \ll 1$ which allows us to neglect this term related to the Doppler shift in Eq. (5). We therefore find the relations

$$F \approx n + \mu \quad \text{and} \quad G \approx \nu . \tag{6}$$

With the result $G \approx \nu$, Eq. (4) reads

$$\left(\frac{\partial}{\partial t} + \Gamma g^2\right)\nu + \frac{2kp}{m}\,\mu = 0 . \tag{7}$$

Since $t \gg 1/\Gamma g^2$ we can neglect in Eq. (7) the time derivative which yields $\nu \approx [-2kp/(m\Gamma g^2)]\,\mu$ and hence Eq. (3) for μ takes the form

$$\frac{\partial\mu}{\partial t} + \left(\frac{2kp}{m}\right)^2 \frac{1}{\Gamma g^2}\,\mu = -\Gamma g^2 F . \tag{8}$$

We assume the natural initial condition $\mu(t = 0) = 0$, that is no coherence between the two ground states at $t = 0$. The solution of the inhomogeneous differential equation (8) of first order then reads

$$\mu(p,t) = -\Gamma g^2 \int_0^t dt'\, K(p, t - t')F(p,t')$$

where we have introduced the kernel

$$K(p,t) \equiv \exp\!\left[-\left(\frac{2kp}{m}\right)^2 \frac{t}{\Gamma g^2}\right]. \tag{9}$$

With the help of Eq. (6), that is $n = F - \mu$, we find

$$n(p,t) = F(p,t) + \Gamma g^2 \int_0^t dt'\, K(p, t - t')\, F(p,t') . \tag{10}$$

Equation (10) is the central point of our paper. It shows that the distribution function n is related to the function F in a local and a nonlocal way in time [18]. In the latter there appears the kernel K whose width δp in momentum

$$\delta p(t) \equiv \frac{m}{2k}\sqrt{\frac{\Gamma g^2}{t}} = \frac{\hbar k}{4\omega_R}\sqrt{\frac{\Gamma g^2}{t}}$$

decreases in time as $t^{-1/2}$. When we compare this width to the width $\delta p_{\rm dif}(t)$ of the function F we find the relation $\delta p(t) \sim \delta p_{\rm dif}(t)/(\omega_R t)$. Therefore, in the time domain $t\omega_R \gg 1$ the width δp of the kernel is much smaller than the width $\delta p_{\rm dif}$ of the function F [19], which allows us to approximate the kernel by a scaled δ function

$$K(p,t) \approx \frac{\hbar k}{4\omega_R}\sqrt{\frac{\pi\Gamma g^2}{t}}\,\delta(p) . \tag{11}$$

Thus the first term in Eq. (10) with the characteristic momentum scale $\delta p_{\rm dif}$ is the distribution function $n_{\rm dif} \equiv F$ for the particles subjected to the diffusion process. The term nonlocal in time and of the form $\delta(p) N_{\rm tr}(t)$ describes the particles trapped in the vicinity of zero momentum. This decomposition is based on the fact that *outside* the trapping area the noncoupled and coupled states are populated almost equally; that is $n_{\rm NC} \approx n_{\rm C}$. Here we therefore have the relations $-\mu = n_{\rm NC} - n_{\rm C} \approx 0$ and $F \approx n + \mu = 2n_{\rm C} \approx n$. In contrast, *inside* the trapping area the population of the noncoupled state is much larger than its counterpart; that is $n_{\rm NC} \gg n_{\rm C}$, and $-\mu \approx n_{\rm NC} \approx n$.

With Eq. (11) we can bring Eq. (10) into the form

$$n(p,t) = n_{\rm dif}(p,t) + \delta(p)\, N_{\rm tr}(t) , \tag{12}$$

where the total number $N_{\rm tr}(t)$ of trapped particles is the convolution [20]

$$N_{\rm tr}(t) \equiv \Gamma g^2 \int_0^t dt'\, \frac{\hbar k}{4\omega_R}\sqrt{\frac{\pi\Gamma g^2}{t - t'}}\, n_{\rm dif}(0,t') . \tag{13}$$

We now interpret this expression for $N_{\rm tr}$ in terms of the statistical approach [3] which treats subrecoil laser cooling as continuous random walks with a trapping process around $p = 0$. Since in the vicinity of zero momentum $\mu \approx -n_{\mathrm{NC}}$ we find from Eq. (8) that

$$\tau_{\mathrm{NC}}(p) \equiv \Gamma g^2 \left(\frac{m}{2kp}\right)^2 \tag{14}$$

is the momentum dependent lifetime [3,12] of the noncoupled state. Hence the kernel $K(p,t) = \exp[-t/\tau_{\mathrm{NC}}(p)]$ gives the decay law of the noncoupled state with momentum p. Integration of K over p therefore yields the probability $w(t)$ for the atom to remain in the trapping state until the time t, provided it was prepared in this state at $t = 0$. The random walk theory [2] refers to this quantity as the waiting function. With the explicit relation (9) for the kernel we find

$$w(t) \equiv \frac{1}{Z}\int dp\, K(p,t) = \frac{1}{Z}\frac{\hbar k}{4\omega_R}\sqrt{\frac{\pi \Gamma g^2}{t}}\,. \tag{15}$$

Since w has to be dimensionless we have introduced the size [3] $Z \equiv \int_{|p|\le p_{\mathrm{trap}}} dp = 2p_{\mathrm{trap}}$ of the trapping state in momentum space [21].

Equation (15) shows that the waiting function $w(t) \propto t^{-1/2}$ of the open Λ system enjoys the same asymptotic time dependence [3] as in the closed Λ system. The reason becomes clear when we recall that the time dependence of $w(t)$, Eq. (15), follows immediately from the p^{-2} dependence of the lifetime, Eq. (14), resulting from the coupling between $|\psi_{\mathrm{NC}}\rangle$ and $|\psi_{\mathrm{C}}\rangle$ states due to the kinetic energy operator [12].

With the identification given by Eq. (15) we rewrite the convolution Eq. (13) as

$$N_{\mathrm{tr}}(t) = \int_0^t w(t - t')\,\Phi(t')\,dt' \tag{16}$$

where

$$\Phi(t) \equiv Z\Gamma g^2 n_{\mathrm{dif}}(0,t) \approx 2\int_{|p|\le p_{\mathrm{trap}}} dp\, \Gamma_g(p,t)\,. \tag{17}$$

In the last step we have used Eq. (1) together with $\Gamma \approx \gamma$ and realized that $\rho_{33} \approx g^2 n_{\mathrm{dif}}$ is almost constant around $p = 0$.

According to Eq. (17), $\Phi(t)$ is the pumping rate into momenta around $p = 0$, that is into the trapping state. We see that an atom reaches this state due to a random event such as spontaneous emission from the upper level and thereby undergoes a recoil from the interval $-\hbar k \le p \le \hbar k$. This allows us to interpret $\Phi(t)dt$ as the probability that between t and $t + dt$ the atom has reached due to the diffusion the trapping state. Therefore the identifications, given by Eqs. (15) and (17), result in the standard stochastic interpretation [2] of the convolution (16) for the trapping state probability.

We now prove that for large times, that is for $t \gg \tau$, the counting rate Γ_{nd} into the nondetected state is given by the waiting-time distribution $P(t) \equiv -\partial w/\partial t \sim t^{-3/2}$. The counting rate $\Gamma_{nd} \equiv -\partial N/\partial t$ follows from the change of the total number N of particles in the ground states $|g_\pm\rangle$. We derive the equation of motion for $N(t)$ by integrating Eq. (2) over p and using the decomposition Eq. (12) which yields

$$\frac{\partial N}{\partial t} = -\frac{N - N_{\mathrm{tr}}}{\tau}\,. \tag{18}$$

Hence the total number of particles is not conserved. This is a consequence of the coefficient $\gamma/\Gamma < 1$ in Eq. (2). Moreover, due to the term N_{tr}/τ, $N(t)$ displays a nonexponential decay.

In the asymptotic regime $t \gg \tau$ we can neglect the time derivative in Eq. (18) and arrive at the relations

$$N(t) \simeq N_{\mathrm{tr}}(t), \qquad \text{that is } N_{\mathrm{dif}} \approx 0\,. \tag{19}$$

In this approximation the number N_{dif} of diffused particles is zero and the distribution function $n_{\mathrm{dif}}(p,t)$ tends to zero as well. Since $\Phi(t)$ is proportional to $n_{\mathrm{dif}}(0,t)$ and n_{dif} tends to zero for $t > \tau$ only times $t' \le \tau$ contribute to the integral Eq. (16). Moreover, when we recognize that $w(t) \sim t^{-1/2}$ is a slowly varying function we can approximate $w(t - t') \simeq w(t)$, factor it out of the integral, and extend the upper limit of integration to infinity. Hence for $t \gg \tau$ the number of trapped particles reads

$$N_{\mathrm{tr}}(t) \simeq w(t)\int_0^\infty dt'\,\Phi(t') \tag{20}$$

and is therefore asymptotically proportional to the waiting function $w(t)$. The convergence [22] of the integral in Eq. (20) is due to the keyhole, that is because of $\gamma_0 \neq 0$.

With the help of Eqs. (19) and (20) the counting rate

$$\Gamma_{nd}(t) \approx -\frac{\partial N_{\mathrm{tr}}}{\partial t} \sim -\frac{\partial w}{\partial t} = P(t) \sim t^{-3/2} \tag{21}$$

in the decay channel describes asymptotically the waiting-time distribution $P(t)$ for the trapped particles.

The relation (12) has a very simple meaning: When $t \gg \tau$ the particles which are not in the trapping state have populated the nondetected level. Therefore the rate of spontaneous fluorescence in the decay channel is proportional to the probability $P(t)$ that the atom leaves the trapping state per unit time. In other words the slowly decreasing power-law tail in the rate of resonance fluorescence into the nondetected state gives a direct evidence of the Lévy statistics in subrecoil laser cooling.

We can also observe these Lévy statistics by monitoring the fluorescence rate $\Gamma_g(t) = 2\int dp\,\Gamma_g(p,t)$ on the transition $|e\rangle \to |g\rangle$. We find from Eq. (1) that $\Gamma_g(t) = \gamma N_e(t)$ is given by the total number $N_e(t)$ of atoms in the excited state. Since $N_e(t) \propto g^2 N_{\mathrm{dif}}(t) \propto -\partial N/\partial t$ it shows the same asymptotic time dependence as $\Gamma_{nd}(t)$, Eq. (21); that is $\Gamma_g(t) \propto P(t)$. We emphasize that this is due to the keyhole; without the decay channel $\Gamma_g(t)$ would be a more complicated function on time.

For an experimental realization we envision a rather monochromatic beam of slow atoms with velocities $v \lesssim 10^3$ cm/s. We observe the dependence of the fluorescence intensity into the level $|nd\rangle$ along the atomic trajectory inside the laser field. The typical signal strength is γ_0/γ of the resonance fluorescence rate on the transition $|e\rangle \rightarrow |g\rangle$. The fact that the time and the space dependence of the angular distribution $I_{nd}(\vec{n}, t)$ of emitted photons factorizes provides some advantages in monitoring the signal. Indeed, the angular distribution $I_g(\vec{n}, t)$ of the fluorescence signal on the transition $|e\rangle \rightarrow |g\rangle$ is determined by the induced dipole moment and does not factorize. It shows a more complicated time dependence [23] than $\Gamma_{nd}(t)$.

In conclusion, we have shown that the waiting-time distribution for the trapped particles in VSCPT governs asymptotically the rate of resonance fluorescence in the decay channel of an open Λ system. The decay constant γ_0 into this channel has to be large enough, $\gamma_0 \gg \omega_R^2/\gamma$, but smaller than γ. Therefore for an experimental realization of the proposed method we need an appropriate level structure. In metastable He the spontaneous decay rate of the upper level to the ground state is negligibly small. However, it may be possible to excite or to ionize selectively the upper level by an additional laser field and monitor the signal in this decay channel. The much richer level structure of Rb atoms which were subjected successfully [24] to subrecoil cooling might even be more suitable for our proposal.

Our results rely on integral characteristics such as the total number of particles and the fluorescence rate of the system. Moreover, the underlying physical picture is rather simple. We are therefore confident that this technique can be extended to higher dimensions.

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