

Entanglement-Assisted Classical Capacity of Noisy Quantum Channels

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Prior entanglement between sender and receiver, which exactly doubles the classical capacity of a noiseless quantum channel, can increase the classical capacity of some noisy quantum channels by an arbitrarily large constant factor depending on the channel, relative to the best known classical capacity achievable without entanglement. The enhancement factor is greatest for very noisy channels, with positive classical capacity but zero quantum capacity. We obtain exact expressions for the entanglement-assisted capacity of depolarizing and erasure channels in d dimensions.

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Prominent among the goals of quantum information theory are understanding entanglement and calculating the several capacities of quantum channels. Physically, a quantum channel can be pictured as the transfer of some quantum system from sender to receiver. If the transfer is intact and undisturbed, the channel is noiseless; if the quantum system interacts en route with some other system, a noisy quantum channel results. Quantum channels can be used to carry classical information, and, if they are not too noisy, to transmit intact quantum states and to share entanglement between remote parties. Unlike classical channels, which are adequately characterized by a single capacity, quantum channels have several distinct capacities. These include a classical capacity C , for transmitting classical information, a quantum capacity Q , for transmitting intact quantum states, a classically assisted quantum capacity Q_2 , for transmitting intact quantum states with the help of a two-way classical side channel, and finally C_E , the *entanglement-assisted classical capacity*, which we define as a quantum channel's capacity for transmitting classical information with the help of unlimited prior pure entanglement between sender and receiver [1]. In most cases, only upper and lower bounds on these capacities are known, not the capacities themselves [2].

Entanglement, e.g., in the form of Einstein-Podolsky-Rosen (EPR) pairs of particles shared between two parties, interacts in subtle ways with other communications resources. By itself, prior entanglement between sender and receiver confers no ability to transmit classical information, nor can it increase the capacity of a classical channel above what it would have been without the entanglement. This follows from the fact that local manipulation of one of two entangled subsystems cannot influence the expectation of any local observable of the other subsystem [3,4]. This is sometimes loosely called the constraint of causality, because its violation would make it possible to send messages into one's past.

On the other hand, it is well known that prior entanglement can enhance the classical capacity of *quantum* channels. In the effect known as superdense coding, dis-

covered by Wiesner [5], the classical capacity of a noiseless quantum channel is doubled by prior entanglement. In other words, $C_E = 2C$ for any noiseless quantum channel. We show that for some channels this enhancement persists, and even increases, as the channel is made more noisy, even after the channel has become so noisy that its quantum capacities Q and Q_2 both vanish, and the channel itself can be simulated by local actions and classical communication between sender and receiver [1].

This is perhaps surprising, since it might seem that any quantum channel that can be classically simulated ought to behave as a classical channel in all respects—in particular, not having its capacity increased by prior entanglement. In fact there is no contradiction, because, as we shall see, even when a quantum channel can be classically simulated, the simulation necessarily involves some amount of forward classical communication from the sender (henceforth “Alice”) to the receiver (“Bob”); and this information is never less than the channel's entanglement-assisted capacity. Thus for any quantum channel, $C \leq C_E \leq FCCC$, where $FCCC$ denotes the forward classical communication cost, i.e., the forward classical capacity needed, in conjunction with other resources, to simulate the quantum channel.

To illustrate these inequalities, consider a specific example, the 2/3-depolarizing qubit channel, which transmits the input qubit intact with probability 1/3 and replaces it by a random qubit with probability 2/3. As is well known, this noisy quantum channel, sometimes referred to as the classical limit of teleportation, can be simulated classically by the following “measure/reprepare” procedure: A third party chooses a random axis R and tells both Alice and Bob. Then Alice measures the input qubit along this axis and tells Bob the one-bit result, after which Bob prepares an output qubit in the same state found by Alice's measurement. Evidently the $FCCC$ of this procedure is 1 bit, but the best known classical capacity of a 2/3-depolarizing channel (realized by encoding 0 and 1 as $|0\rangle$ and $|1\rangle$ on the input side and measuring in the same basis on the output side) is about 0.0817 bits,

the capacity of a classical binary symmetric channel of crossover probability $1/3$. As we shall show, the C_E of the $2/3$ -depolarizing channel is about 0.2075 bits, more than twice the unassisted value, but still safely less than the 1 bit forward classical cost of simulating the channel by measure/reprepare, which we denote $FCCC_{MR}$.

Suppose we wished to simulate not a $2/3$ -depolarizing channel, but a $5/6$ -depolarizing channel. Clearly, this could be done by simulating the $2/3$ -depolarizing channel, then further depolarizing its output. But a more economical simulation would be for Alice to send her one-bit measurement result to Bob not through a noiseless classical channel but through a noisy classical channel of correspondingly lesser capacity. If she sent it through a binary symmetric channel of randomization probability $1/2$ (equivalent to a crossover probability $1/4$), the $5/6$ -depolarizing channel would have been simulated at an $FCCC_{MR}$ of only $1 - H_2(1/4) \approx 0.1887$ bits per channel use, where H_2 is the binary Shannon entropy $H_2(p) = -p \log_2 p - (1 - p) \log_2(1 - p)$. This is of course greater than the $5/6$ -depolarizing channel's best known classical capacity of $1 - H_2(5/12) \approx 0.02013$. The $5/6$ -depolarizing channel's entanglement-assisted capacity must lie between these two bounds.

We now develop these ideas further to obtain an exact expression for C_E for an important class of channels, the d -dimensional depolarizing channel $\mathcal{D}_x^{(d)}$ of depolarization probability x . This is the channel that transmits a d -state quantum system intact with probability $1 - x$ and randomizes its state with probability x . We show that in the high-depolarization limit $x \rightarrow 1$ this channel's entanglement-assisted capacity is $(d + 1)$ -fold higher than the best known lower bound on the classical capacity of the same channel without prior entanglement. This lower bound, the "one-shot" classical capacity C_1 , is defined as the maximum classical information that can be sent through a *single* use of the channel, without prior entanglement, by an optimal choice of source states at the channel input and an optimal measurement at the channel output. For this highly symmetric channel, this optimum can be achieved by assigning equal probability $1/d$ to each state of an arbitrary orthonormal basis $\{|0\rangle, |1\rangle, \dots, |d - 1\rangle\}$ at the channel input, and performing a complete von Neumann measurement in the same basis at the channel output. This causes the quantum channel to behave as a d -ary symmetric classical channel of randomization probability x , giving a capacity

$$C_1(\mathcal{D}_x^{(d)}) = \log_2 d - H_d\left(1 - x \frac{(d - 1)}{d}\right), \quad (1)$$

where $H_d(p) = -p \log_2(p) - (1 - p) \log_2[(1 - p)/(d - 1)]$ is the Shannon entropy of a d -ary distribution consisting of one element of probability p and $d - 1$ elements each of probability $(1 - p)/(d - 1)$. This input ensemble is known to be optimal, for a one-shot use of the channel, because it saturates the Holevo bound

$C_1 \leq \log_2 d - \overline{S(\rho_i)}$, on the one-shot capacity [6], where $\overline{S(\rho_i)}$ is the average von Neumann entropy of the output states ρ_i .

Similarly, it is easy to generalize the measure/reprepare construction to show that a d -dimensional depolarizing channel can be simulated classically whenever $x \geq d/(d + 1)$, at a cost

$$FCCC_{MR}(\mathcal{D}_x^{(d)}) = \log_2(d) - H_d\{d - x[d - (1/d)]\}. \quad (2)$$

The simulation is performed by having Alice measure in a preagreed random basis, send Bob the result through a d -ary symmetric noisy classical channel, after which he reprepares an output state in the same basis. Figure 1 compares the definitions of asymptotic capacity C and one-shot capacity C_1 , and illustrates the measure/reprepare technique for simulating some noisy quantum channels classically.

So far, we have given only lower and upper bounds on C_E , without calculating C_E itself. To do so, we use

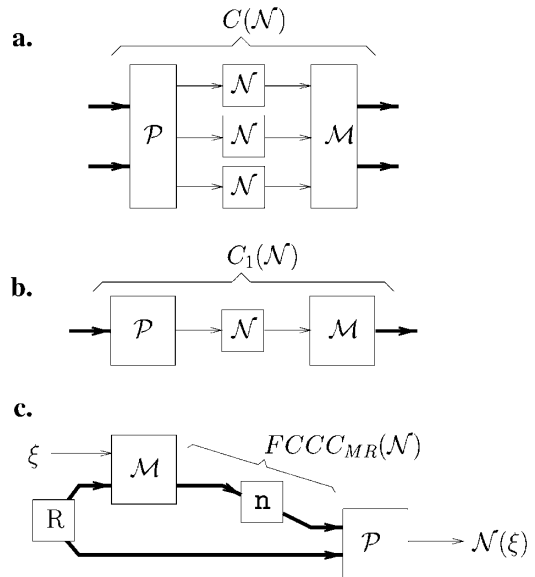


FIG. 1. (a) The classical capacity C of a quantum channel \mathcal{N} is the optimal asymptotic input:output mutual information per channel use achievable by a preparator \mathcal{P} mapping classical inputs (thick lines) to a possibly entangled input state supplied to multiple instances of the quantum channel, and a collective measurement \mathcal{M} mapping the possibly entangled output state back to classical data. (b) The one-shot classical capacity C_1 is the maximum classical mutual information between input preparations and output measurement results for a single use of the channel. (c) Some quantum channels \mathcal{N} , acting on an unknown quantum input ξ , can be simulated classically by having Alice measure the quantum input in a common random basis R , send Bob the result through a noiseless or noisy classical channel n , after which he uses it to reprepare an approximation $\mathcal{N}(\xi)$ to the quantum input ξ at the channel output. The forward classical communication cost ($FCCC_{MR}$) of this simulation is the classical (Shannon) capacity of the internal classical channel n . Allowing n to be noisy makes \mathcal{N} more noisy, but reduces $FCCC_{MR}$.

modified versions of the well-known superdense coding [5] and teleportation [3] protocols to obtain tighter lower and upper bounds, respectively, which in the case of depolarizing and erasure channels coincide, thereby establishing C_E exactly for these channels. We treat the case where \mathcal{N} is a generalized depolarizing channel $\mathcal{D}_x^{(d)}$ first.

Clearly, C_E for any noisy channel \mathcal{N} can be lower-bounded by the entanglement-assisted capacity via a particular protocol, namely, superdense coding with the noisy quantum channel \mathcal{N} substituted for the usual noiseless return path for Alice's half of a shared maximally entangled EPR state Ψ . This version of superdense coding is illustrated in Fig. 2a, and we shall use $C_{Sd}(\mathcal{N})$ to denote the entanglement-assisted capacity of \mathcal{N} via this protocol. Conversely (Fig. 2b), $C_E(\mathcal{N})$ can be upper-bounded by the forward classical communication cost of simulating \mathcal{N} , not by measure/prepare, but by a version of teleportation in which the requisite amount of noise is introduced by substituting a noisy classical channel \mathcal{N} for the usual noiseless classical arm of the teleportation procedure (the classical channel \mathcal{N} operates on a d^2 -letter classical alphabet, in contrast to the d -letter alphabet used by the channel \mathcal{N} in the measure/prepare simulation of Fig. 1c). This upper bound follows from the fact that even in the presence of prior shared entanglement, the FCCC of simulating a quantum channel cannot be less than its classical capacity; otherwise a violation of causality would occur. Whenever a quantum channel \mathcal{N} can be simulated by teleportation with a noisy classical arm, we use $FCCC_{Tp}(\mathcal{N})$ to denote the forward classical communication cost of doing so.

In the case of depolarizing channels, the two bounds coincide, because of the readily verified fact that superdense coding and teleportation map each x -depolarizing

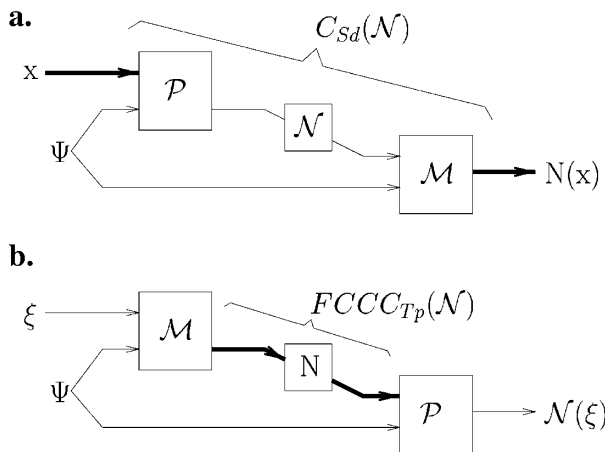


FIG. 2. (a) By using a noisy quantum channel \mathcal{N} in the protocol for superdense coding, one obtains a lower bound C_{Sd} on its entanglement-assisted capacity $C_E(\mathcal{N})$. (b) By using a noisy classical channel \mathcal{N} in the protocol for teleportation to simulate a quantum channel \mathcal{N} , one obtains an upper bound $FCCC_{Tp}$ on $C_E(\mathcal{N})$. When these two bounds coincide, they give $C_E(\mathcal{N})$ exactly.

d -dimensional quantum channel into an x -randomizing d^2 -ary symmetric classical channel and vice versa. Thus, for all depolarizing channels $\mathcal{D}_x^{(d)}$,

$$C_E = C_{Sd} = FCCC_{Tp} = 2 \log_2 d - H_{d^2} \left(1 - x \frac{d^2 - 1}{d^2} \right). \quad (3)$$

From Eqs. (1) and (3), it can be seen that in the high-noise limit $x \rightarrow 1$, the enhancement factor C_E/C_1 approaches $d + 1$. Thus, prior entanglement can increase classical capacity by an arbitrarily large factor. For large d , $C_E/C_1 \approx 2$ for most x , rising sharply near $x = 1$.

We now turn to the quantum erasure channel [7], which is unusual among noisy quantum channels in that its capacities C , Q , and Q_2 are known exactly [2]. A quantum erasure channel transmits its d -dimensional input state intact with probability $1 - x$, and with probability x replaces the input by a unique $(d + 1)$ th state, called an erasure symbol, orthogonal to all the input states. If the channels \mathcal{N} and \mathcal{N} in Fig. 2 are taken to be, respectively, a d -dimensional quantum erasure channel and a d^2 -dimensional classical erasure channel, the superdense coding and teleportation bounds can again easily be shown to coincide, providing an entanglement-assisted capacity $C_E = 2(1 - x) \log d$, exactly twice the erasure channel's ordinary classical capacity.

Figure 3 (left) shows all the capacities of the quantum erasure channel. These capacities are of interest not only in their own right, but also because they upperbound the corresponding capacities of the depolarizing channel, since a quantum erasure channel can simulate a depolarizing channel by having the receiver substitute a fully depolarized state for every erasure symbol he receives.

Returning to the depolarizing channel, we are in the peculiar position of knowing its entanglement-assisted classical capacity C_E without knowing its ordinary unassisted classical capacity C . The latter is generally believed to be equal to the one-shot unassisted capacity C_1 , but the possibility cannot be excluded that a higher capacity might be achieved asymptotically by supplying entangled inputs to multiple instances of the channel (this cannot occur for C_E , where any larger capacity would exceed $FCCC_{Tp}$, violating causality). The range of possible values for the depolarizing channel's unassisted classical capacity

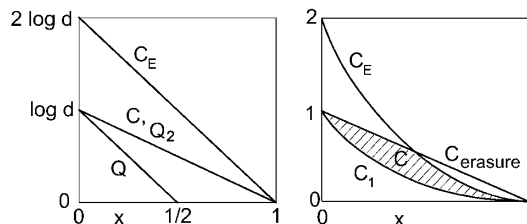


FIG. 3. Left: Capacities of the quantum erasure channel. Right: Bounds on the asymptotic classical capacity C of the qubit depolarizing channel.

C is bounded below by its known C_1 , and above by its known C_E and by the known unassisted classical capacity $(1-x)\log d$ of the quantum erasure channel. Figure 3 (right) shows these bounds for the qubit case $d = 2$.

Although the depolarizing channel's unassisted capacity C remains unknown in absolute terms for all d , the bounds $C_1 \leq C \leq (1-x)\log d$ become increasingly tight *relative* to C as $d \rightarrow \infty$, because, as can readily be verified, the difference between the bounds approaches $H_2(x)$ in this limit. Similarly, the depolarizing channel's unassisted *quantum* capacity Q is upperbounded by the erasure channel's quantum capacity, $\max\{0, 1-2x\}\log d$, and lowerbounded by the depolarizing channel's quantum capacity via random hashing [8], $\log d - S[[\mathcal{N} \otimes I](\Psi)]$. Here $[\mathcal{N} \otimes I](\Psi)$ is the mixed state formed by sending half a maximally entangled $d \otimes d$ pair Ψ through the noisy channel. Again the difference between the bounds approaches $H_2(x)$ as $d \rightarrow \infty$.

The equality between $FCCC_{Tp}$ and C_{Sd} , which makes C_E exactly calculable for depolarizing channels, holds for all "Bell-diagonal" channels [8], those that commute with superdense coding and teleportation, so that $Tp[Sd(\mathcal{N})] = \mathcal{N}$ [9]. For example, the qubit dephasing channel, which subjects its input to a σ_z Pauli rotation with probability $x/2$, has $C = 1$ independent of x , while $C_E = 2 - H_2(x/2)$. For other channels, it can be shown [10] that

$$C_E = \max_{\Psi} \{S(\rho) + S(\mathcal{N}(\rho)) - S([\mathcal{N} \otimes I](\Psi))\}, \quad (4)$$

where Ψ is a bipartite pure state in $d \otimes d$, and ρ is its partial trace over the second party. This capacity can be achieved asymptotically by applying superdense coding to a Schumacher-compressed version of $\rho^{\otimes n}$ for large n , and evaluating the resulting classical capacity by Holevo's formula [6]; that C_E can be no higher can be shown [10] using Holevo's formula and the strong subadditivity property of quantum entropy.

A channel's entanglement-assisted *quantum* capacity Q_E may be defined as its maximum rate for transmitting intact qubits with the help of prior entanglement but no classical communication. By teleportation and superdense coding, $Q_E = C_E/2$ for all channels. Naturally, Q_E upperbounds the unassisted quantum capacity Q , but in most instances, e.g., the depolarizing channel, tighter upper bounds are known.

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