"One-Sided" Log-Normal Distribution of Conductances for a Disordered Quantum Wire

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We develop a simple systematic method, valid for *all* strengths of disorder, to obtain analytically the full distribution of the conductance $P(g)$ for a quasi-one-dimensional wire in the absence of electron-electron interactions. We show that in the crossover region between the metallic and insulating regimes $P(g)$ is highly asymmetric, given by an essentially "one-sided" log-normal distribution. For larger disorder, the tail of the log-normal distribution for $g > 1$ is cut off by a Gaussian.

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Since the discovery of the absence of self-averaging in mesoscopic disordered systems [1], the study of the full distribution of conductance has attracted a lot of attention [2]. In particular, while the metallic regime is well described by a Gaussian distribution, the moments of the conductance fluctuations become of the same order of magnitude as the average conductance on approaching the localized regime. In such cases the average value becomes insufficient in describing properties of disordered conductors and the full distribution must be considered. Recently, numerical support for the existence of a new *universal* distribution at the metal-insulator transition [3], a broad distribution of the critical conductance at the integer quantum Hall transition [4], as well as the expected multifractal properties associated with the critical regime [5], have increased the interest in the conductance distribution in the intermediate regime, between the well-studied universal conductance fluctuations in the metallic limit and the log-normal distribution in the deeply insulating limit. However, even for a quasi-one-dimensional (1D) system where there is only a smooth crossover between the metallic and insulating regimes, there is no analytic result available for the conductance distribution in the crossover regime. So far only the first two moments have been obtained for all strengths of disorder [6], using the 1D supersymmetric nonlinear σ model [7]. This model has been shown [8] to be equivalent, in the thick wire or quasi-1D limit, to the Dorokov-Mello-Pereyra-Kumar (DMPK) equation [9] which describes the evolution of the distribution of the transmission eigenvalues with increasing wire length.

In this work we develop a simple systematic method to evaluate directly the full distribution of conductance for a thick quasi-1D wire (mean free path $l \gg$ the width), starting from the solution of the DMPK equation. The main result of the paper is that although there is no phase transition in quasi-one-dimension, the crossover region between metallic and insulating regimes is highly nontrivial, and shows a remarkable "one-sided" log-normal distribution. Recent numerical studies of a quasi-1D system in the quantum Hall regime have shown highly asymmetric log-normal distributions in the crossover region [10]. We expect similar qualitative features to exist in the critical regimes in higher dimensions as well. Indeed, numerical studies near the integer quantum Hall transition in two dimensions as well as the Anderson transition in three dimensions also point to asymmetric distributions of the critical conductance [11]. In addition, we predict that even the insulating regime should have a sharp cutoff in its log-normal tail near the (dimensionless) conductance $g \sim 1$. In particular, we show that the conductance distribution in the insulating regime (in the absence of time reversal symmetry) has the form

$$P[\ln(g)] \approx \begin{cases} \sqrt{\frac{x_1 \sinh 2x_1}{1-g}}\, e^{-\Gamma x_1^2}, & g < 1, \\ \sqrt{2}\, g e^{-a(g-1)^2}, & g \geq 1, \end{cases} \qquad (1)$$

where $x_1 = \cosh^{-1}(1/\sqrt{g})$ and the parameter $\Gamma = \xi/L$, where $\xi = Nl$ is the quasi-1D localization length, N is the number of transmission channels, and $L \gg l$ is the length of the conductor. The parameter a is the value of F'' given in (10) evaluated at $x_2 = 2/\pi\Gamma$, and tends to $\frac{3}{8}\exp[8/\pi\Gamma]$ for $\Gamma \ll 1$ in insulators. Note that for $g \ll 1$, $x_1 \sim \ln(2/\sqrt{g})$, and the distribution is log normal, centered at $-\ln g = 1/\Gamma$. However, for $g > 1$, the tail is cut off by an exponential function over an exponentially narrow scale in $1/\Gamma$. The results for two different values of Γ, 0.7 and 0.2, are plotted in Fig. 1. The main point is that for very strong disorder, the typical value of g is much smaller than unity, so the peak of the distribution $P[\ln(g)]$ is very far away from $g \sim 1$. In this case the exponential cutoff at $g \sim 1$ is less relevant. However, even for large disorder, a sharp cutoff in the tail for $g > 1$ always exists, as shown for $\Gamma = 0.2$ in Fig. 1. At intermediate strength of disorder, the peak of the distribution is close to the cutoff, and the distribution becomes highly asymmetric. In particular, near the crossover between metallic and insulating behavior, the peak is at $g \sim 1$, and we obtain a one-sided log-normal distribution, as shown for $\Gamma = 0.7$ in Fig 1.

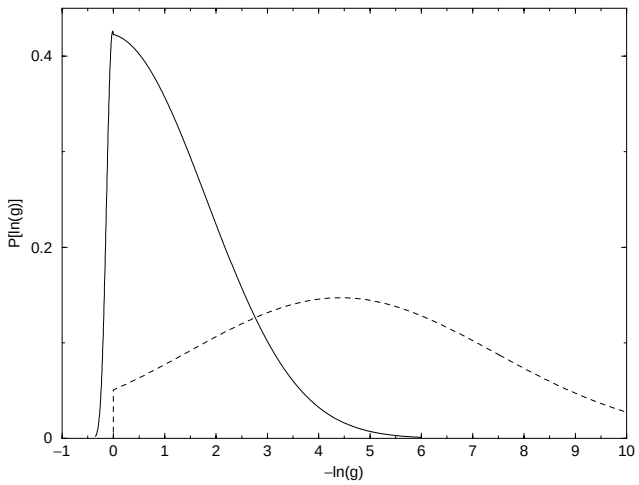

FIG. 1. Log-normal distribution of conductance $P[\ln(g)]$ given by Eq. (1) in the insulating regime for two strengths of disorder, $\Gamma = 0.2$ (dashed line) and $\Gamma = 0.7$ (solid line).

As a check of the scope and validity of the method developed here, we obtain the exact universal conductance fluctuation in the weakly disordered metallic regime with the expected Gaussian distribution as well as the correct mean and variance of the log-normal distribution in the strongly disordered localized regime [2] within the same unified framework. Systematic corrections as a function of disorder can be obtained from both the metallic and the insulating limits. Note however that the analytic results presented here near the crossover regime are only semiquantitative, due to the approximate analytical evaluations of certain integrals. A more quantitative result is possible based on numerical evaluations of these integrals.

We first briefly outline the method. The probability distribution $p(\lambda)$ of the N variables λ_i, related to the transmission eigenvalues T_i of an N-channel quasi-1D wire by $\lambda_i = (1 - T_i)/T_i$, satisfies the well-known DMPK equation [9]. The solution for this equation can be written in the general form [12] $p(\lambda) \propto \exp[-\beta H(\lambda)]$, where $H(\lambda)$ may be interpreted as the Hamiltonian function of N classical charges at positions λ_i. The symmetry parameter $\beta = 1, 2$, or 4 depending on the symmetry of the ensemble [2]. The Hamiltonian depends on the parameters L, N, and l only in the combination $\Gamma = Nl/L$. We will consider the quasi-1D limit where both N and L approach infinity keeping Γ fixed. The dimensionless conductance in terms of λ_i is given by $g = \sum_i^N \frac{1}{1+\lambda_i}$ [13]. The distribution of conductance can therefore be written as

$$P(g) = \frac{1}{Z} \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} \int_0^{\infty} \prod_{i=1}^{N} d\lambda_i \exp\left[i\tau\left(g - \sum_i^N \frac{1}{1 + \lambda_i}\right) - \beta H \right], \tag{2}$$

where Z is a normalization factor. In the metallic regime $g \gg 1$, the λ_i are all very close to each other so that a continuum description can be used with a density of λ finite between zero and an upper cutoff given by the normalization condition. This approximation describes the universal conductance fluctuations in the metallic regime [14]. In the deeply insulating regime, on the other hand, all λ_i are exponentially large and separated exponentially from each other, and the conductance is dominated by the smallest eigenvalue. This approximation describes the log-normal distribution in the deeply insulating regime [15]. It is clear however that none of the above descriptions can be used in the crossover regime, where the smallest eigenvalue is neither zero, nor exponentially large. Nevertheless, it turns out that it is possible to combine the essential features of the two descriptions and develop a simple and systematic procedure to study the conductance distribution at intermediate regimes. For simplicity, we will discuss the case $\beta = 2$ only. The basic idea is the following:

(1) We first separate out the lowest eigenvalue λ_1 and treat the rest as a continuum with a lower bound at $\lambda_2 > \lambda_1$. Note that this approximation can be systematically improved by separating out the lowest $n > 1$ eigenvalues and treating the rest as a continuum.

(2) The continuum part can be written as a functional integration on the generalized density $\rho(\lambda)$, and the distribution (2) can be rewritten as

$$P(g) = \frac{1}{Z} \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} e^{i\tau g} \int_0^{\infty} d\lambda_1 \int_{\lambda_1}^{\infty} d\lambda_2 \int D[\rho(\lambda)] \exp\{-F[\lambda_1, \lambda_2; \rho(\lambda); \tau]\}. \tag{3}$$

Here the "free energy"

$$F[\lambda_1, \lambda_2; \rho(\lambda); \tau] = \beta H[\lambda_1, \lambda_2; \rho(\lambda)] + i\tau\left[\frac{1}{1 + \lambda_1} + \int_{\lambda_2}^{b} d\lambda \frac{\rho(\lambda)}{1 + \lambda} \right] \tag{4}$$

contains the "edge" separating out λ_1 as well as the "source" terms proportional to τ, plus the continuum version of the Hamiltonian of the form $H = \sum_{i<j}^N u(\lambda_i, \lambda_j) + \sum_i^N V(\lambda_i)$. The upper limit b is given by the number conservation $\int_{\lambda_2}^{b} d\lambda\, \rho(\lambda) = N - 1$.

(3) We obtain the density by minimizing the free energy with respect to $\rho(\lambda)$, keeping λ_1 and λ_2 fixed. This gives

$$\int_0^{\infty} d\lambda' u(\lambda + \lambda_2, \lambda' + \lambda_2) \rho_{\mathrm{sp}}(\lambda + \lambda_2) = 2V_{\mathrm{tot}}(\lambda + \lambda_2), \tag{5}$$

where we have shifted the lower limit to zero, and $V_{\rm tot}(\lambda) = V(\lambda) + \frac{i\tau/\beta}{1+\lambda} + u(\lambda_1, \lambda)$. After taking a derivative on both sides of (5), the kernel can be inverted to obtain the saddle point density $\rho_{\rm sp}(\lambda)$.

(4) From the density, we obtain the saddle point free energy

$$F_{\rm sp} = \frac{\beta}{2} \int_{\lambda_2}^{b} d\lambda\, V_{\rm tot}(\lambda)\rho_{\rm sp}(\lambda) + \beta V(\lambda_1) + \frac{i\tau}{1+\lambda_1}. \tag{6}$$

(5) Since $V_{\rm tot}$ and therefore $\rho_{\rm sp}$ are both linear in τ, the free energy is quadratic in τ and can be written in the form $F_{\rm sp} = F^0 + (i\tau)F' + \frac{(i\tau)^2}{2}F''$. The integral over τ in Eq. (3) can then be done exactly. The result is

$$P(g) = \frac{1}{Z} \int_0^\infty d\lambda_1 \int_{\lambda_1}^\infty d\lambda_2\, e^{-S}; \qquad S = -\frac{(g - F')^2}{2F''} + F^0. \tag{7}$$

(6) At this point, the integrals over λ_1 and λ_2 can be evaluated numerically. Instead, we use saddle point approximation to do the integrals in order to obtain an analytic expression for $P(g)$. Solving for $\frac{\partial S}{\partial \lambda_i} = 0$ for $i = 1, 2$ to determine the saddle point values of λ_1 and λ_2, we obtain the distribution as a function of the conductance g, in terms of the parameter Γ.

In the above approach, if we set both λ_1 and λ_2 equal to zero, we obtain the correct universal value $2/15\beta$ for the variance of g. This is consistent with the picture that in the metallic regime the eigenvalue density can be treated as a continuum from zero to an upper cutoff b. As disorder is increased beyond the metallic regime, this picture starts to break down. In particular, the smallest eigenvalue is pushed to a finite distance from zero depending on the strength Γ, so that the continuum picture at the edge no longer holds. The correction to the metallic behavior is captured in the present approach by evaluating the shifts in λ_1 and λ_2 within a variational scheme. Since the insulating regime is dominated by the smallest eigenvalue, this approach clearly captures the correct insulating behavior. In the crossover regime, both the separation of the smallest eigenvalue as well as the rest of the continuum become important. Note that if more accuracy is needed, one can in principle separate out more than one eigenvalue.

We now give some details. From the exact solution of the DMPK equation, the two and one body terms in the Hamiltonian of (2) are known to be [12]

$$u(\lambda, \lambda') = -\frac{1}{2} \ln|(\lambda - \lambda')[x^2(\lambda) - x^2(\lambda')]|; \qquad V(\lambda) = \frac{\Gamma}{2} x^2(\lambda), \tag{8}$$

where $x(\lambda) = \sinh^{-1}\sqrt{\lambda}$. Note that the difference $\Delta u = u(\lambda + \lambda_2, \lambda' + \lambda_2) - u(\lambda, \lambda')$ is negligible in the insulating regime and is a small correction proportional to λ_2 in the metallic regime. Therefore to a first approximation, the shifted kernel $u(\lambda + \lambda_2, \lambda' + \lambda_2)$ can be replaced by the unshifted kernel $u(\lambda, \lambda')$. One can then use the saddle point density obtained from the unshifted kernel to calculate the correction due to the change in the kernel from the shift. This can be rewritten as an additional term $\lambda_2 V_2 + V_{\rm tot} = V_{\rm eff}$ with an unshifted kernel in Eq. (5). The unshifted kernel can then be inverted to give the saddle point density

$$\rho_{\rm sp}(\lambda + \lambda_2) = \frac{1}{\lambda(1+\lambda)} \int_{-\infty}^{\infty} d\lambda' K^{-1}[x(\lambda) - x(\lambda')] \frac{d}{d\lambda'} V_{\rm eff}(|\lambda'| + \lambda_2), \tag{9}$$

where the inverse of the unshifted kernel is $K^{-1}(t) = -(1/2\pi^2)\int_0^\infty dq \sin(qt)(1 - e^{-\pi q})$. The condition $\rho_{\rm sp}(\lambda) \geq 0$ for all λ in Eq. (9) requires $\lambda_2 - \lambda_1 > \lambda_c = (2/\Gamma\pi)^2$. The free energy can now be obtained from Eq. (6). The integrals for F'' can be done exactly. In terms of the variables x_1, x_2, defined as $\sinh^2 x_i = \lambda_i, i = 1, 2$, we get

$$F''(x_2) = \frac{1}{\sinh^2 2x_2}\left[-\frac{1}{3} + \frac{1}{4x_2^2} - \frac{1}{\sinh^2 2x_2}\right]. \tag{10}$$

The integrals for F' and F^0 can be evaluated analytically in two limits. For $x_2 \ll 1$,

$$F' \approx \Gamma - b_1\Gamma x_2^2 + \frac{32}{\pi^3}\sqrt{x_2^2 - x_1^2}, \qquad F^0 \approx \frac{3\pi^2}{8}\Gamma^2 x_2^2 - 2\pi\Gamma\sqrt{x_2^2 - x_1^2} + \frac{3}{2}\ln(x_2^2 - x_1^2) - \ln x_1, \tag{11}$$

where $b_1 \approx 0.89$. In the other limit $x_2 \gg 1$,

$$F' \approx \frac{1}{\cosh^2 x_1}, \qquad F^0 \approx \Gamma x_1^2 - \frac{1}{2}\ln(x_1 \sinh 2x_1) + \frac{1}{3}\Gamma^2 x_2^3 - \Gamma x_2^2 + x_2. \tag{12}$$

In the metallic regime, $\Gamma \gg 1$, and x_2 can be very small. Then Eqs. (10) and (11) give the correct mean conductance $\langle g\rangle = \Gamma$ and variance $\mathrm{var}(g) = 1/15$. When $\Gamma < 1$, $\lambda_2 - \lambda_1 > \lambda_c$ requires $x_2 \gg 1$. In this case the limit $x_1 \gg 1$ corresponds to the insulating limit, but the limit $x_1 \ll 1$ corresponds to the intermediate case close to the crossover regime. We therefore study this regime within a saddle point approximation for the integrals (7).

The condition $\frac{\partial S}{\partial x_1} = 0$ has the solution $\cosh x_1^{\rm sp} = 1/\sqrt{g}$, while the condition $\frac{\partial S}{\partial x_2} = 0$ has the solution $x_2^{\rm sp} = 1/\Gamma$. This leads to the saddle point result $S^{\rm sp}$, to which the contributions from the fluctuations $S^{\rm fl} = \ln|\partial^2 S/\partial x_1^2|$ have to be added, leading to Eq. (1). In the deeply insulating regime $\Gamma \ll 1$, the above expression leads to the known mean and variance $\langle\ln(1/g)\rangle = \mathrm{var}[\ln(g)]/2 = 1/\Gamma$. However, note that since $\cosh x_1 \geq 1$ the saddle

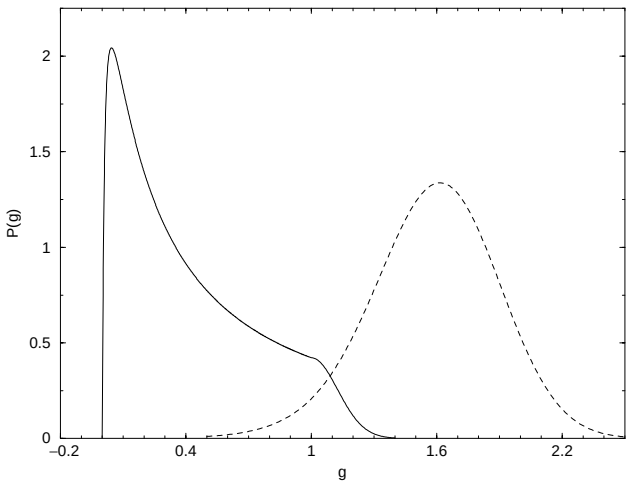

FIG. 2. Distribution of conductance $P(g)$ on the insulating and metallic sides of the crossover regime ($\Gamma \sim 1$) for two strengths of disorder, $\Gamma = 0.7$ (solid line) and $\Gamma = 1.6$ (dashed line).

point solution exists only for $g \leq 1$. For $g > 1$, the x_1 and x_2 integrals are dominated by the boundary values at $x_1 = 0$ and $x_2 = 2/\pi\Gamma$, which has been incorporated in (1). In Fig. 2 we show $P(g)$ as obtained from Eq. (1) for $\Gamma = 0.7$. The skewed shape and the exponential drop at $g \sim 1$ compare well with numerical results of [10] for a slightly smaller value of $\Gamma = 0.5$. The difference in the Γ values simulates to some extent the correction terms to Eq. (1) expected for values of Γ approaching unity. Also shown in Fig. 2 is the result of a numerical integration of Eq. (7) using Eq. (11) for $\Gamma = 1.6$ in the metallic regime. The Gaussian shape of $P(g)$ obtained for this rather small value of Γ is in very good agreement with the results of [10].

We note that according to the relation $g = \sum \frac{1}{1+\lambda_i}$ any non-negligible $P(g \sim 1)$ comes from the possibility that the smallest eigenvalue λ_1 can be close to the origin. However, given $\lambda_1 \ll 1$, the logarithmic repulsion between eigenvalues generated from (8) forces the rest of the eigenvalues exponentially far when $\Gamma \ll 1$, so $P(g > 1)$ is cut off sharply. Since these arguments are quite general, we expect qualitatively similar features in higher dimensions as well, which should have important consequences for the universal conductance distribution in the critical regime.

To summarize, we calculated the distribution of conductances $P(g)$ for a quasi-1D disordered system using known results for the DMPK equation. In this case $P(g)$ depends only on one parameter $\Gamma = \xi/L$, where ξ is the localization length. In the crossover regime $\xi/L \sim 1$, we find that $P(g)$ is given by a one-sided log-normal distribution, cut off by a Gaussian tail on the metallic side ($g > 1$). We believe that this behavior could be generic for $P(g)$ in the transition regime even in higher dimensions, provided the average of g at the transition or crossover region is of order unity. Our results cannot be directly compared to the work of [16] in $d = 2 + \epsilon$ dimensions, because for $\epsilon \ll 1$, $\langle g \rangle = 1/\epsilon$ is large, and the bulk of $P(g)$ is located deep in the metallic regime. As proposed in [17], the center of $P(g)$ is then Gaussian, with power law tails $\propto g^{-2/\epsilon}$. The latter results are peculiar to the behavior in $2 + \epsilon$ dimensions. One should keep in mind that the DMPK approach does not contain the effects of wave function correlations in the transverse direction, which are expected to be important in higher dimensions. Nonetheless, the similarities of the shape of $P(g)$ in the crossover regime obtained here with the numerically determined $P(g)$ in 3D at the critical point [11] appear to suggest that the generic behavior of $P(g)$ is that of a log-normal distribution for $g < 1$ combined with a Gaussian cut off for $g > 1$.

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