Steady State Thermoelectric Field-Reversed Configurations

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It is shown that the cross-field thermoelectric force of magnetized plasmas can maintain field-reversed configurations against resistive diffusion, resulting in a steady state device attractive for thermonuclear fusion. If a peaked radial temperature profile is maintained, the thermoelectric force is in the opposite direction to the usual resistive friction, thus maintaining the field configuration. The field maintenance is tantamount to dynamo action, operating even in two dimensions. We show that a steady state device can be made by simply heating the O-point: no external electric fields or particle sources are needed. The feasibility of this scheme for fusion is discussed.

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The field-reversed configuration (FRC) is attractive as a device to contain thermonuclear plasmas because of its relative simplicity and because it lends itself to direct conversion of the fusion products [1]. The configuration has a closed, purely poloidal magnetic field and is toroidally symmetric. The plasma pressure is contained by the encircling magnetic pressure and magnetic tension. All the currents generating the field come from plasma diamagnetism.

The FRC has two drawbacks: The magnetic field diffuses and decays on resistive time scales if it is not maintained by an external toroidal electric field; hence, the device is not steady state. Second, the system may be unstable to MHD interchange and kink modes. The latter is a topic of intensive research; in this paper we address the steady state issue.

Specifically, we show in this paper that the cross-field thermoelectric force [2] can maintain an FRC in steady state as long as an electron temperature gradient is maintained by external heating. The thermoelectric force, also referred to as the Nersnt effect, is a force that the electron fluid feels on account of the fact that the electron-ion collision frequency decreases with increasing relative speed, v_R, as v_R^{-3}. The physics underlying the cross-field thermoelectric force [3] and how it can maintain the magnetic field is schematized in Fig. 1. In the figure, the magnetic field points out of the page and there is a plasma temperature gradient across the field. Consider the middle portion of the figure: there is a diamagnetic flow of electrons, from left to right in the picture, on account of the difference in gyrospeeds (for simplicity, we assume constant density and immobile ions). The usual friction force arises because of the friction of this diamagnetic flow on the background ions. This is the origin of the cross-field resistivity, and this is what causes the magnetic flux to diffuse at a rate proportional to $\eta_\perp \nabla T_e$. Note that this frictional force on the electrons is from right to left. Consider now that cold particles (in the bottom portion of the figure) collide more efficiently than the hot ones, due to the v_R^{-3} effect. In that case, if we consider again the middle portion of Fig. 1, we note that the part of the electron fluid moving to the left feels more friction than the part moving to the right. Thus, the differential friction on the electron fluid due to the v_R^{-3} effect is from left-to-right. This is in the *opposite* direction to the "normal" friction. Furthermore, this effect is also proportional to $\eta_\perp \nabla T_e$. If a density gradient is included in the above accounting, the normal friction is proportional to $\nabla(nT_e)$, but the countervailing thermoelectric friction is still only proportional to ∇T_e. Thus, by balancing $\nabla(nT_e)$ with a countervailing term proportional only to ∇T_e, unique steady state profiles can be found which are characterized by there being no net friction on the electron fluid. Hence, there is no diffusion of the magnetic field.

That the thermoelectric effect can be used to maintain plasma current has been recognized before. Sakharov [4] evaluated this effect in the transport equilibrium of a

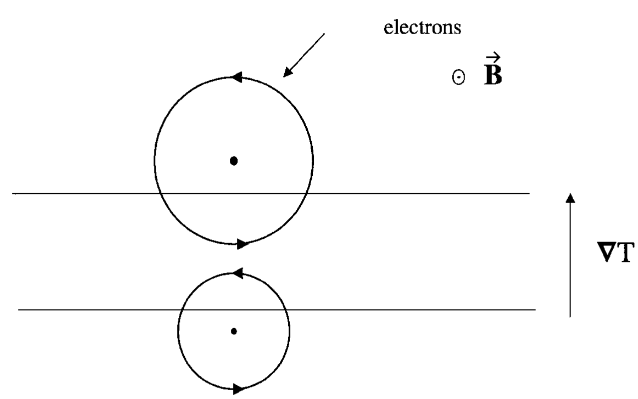

FIG. 1. The origin of the thermoelectric force. The differential friction between hot and cold gyro-orbits is in a direction opposite to the normal friction.

cylindrical plasma column with an axial magnetic field. Ahlborn [5] studied such a system in more detail. A suggestion to use this effect for an FRC, along the lines of this paper, has also been made by Steinhauer [6].

In what follows, we use the Braginskii [2] equations to present an analytic theory of this effect. The transport equations for a steady state MHD configuration can be written as

$$\vec{\nabla}\cdot(n\vec{u}) = S\,, \tag{1}$$

$$\vec{\nabla}(nT_e + nT_i) = \vec{j}\times\vec{B}\,, \tag{2}$$

$$\begin{aligned}\vec{E}_0 - \vec{\nabla}\Phi = -\vec{v}\times\vec{B} - (ne)^{-1}\vec{\nabla}(nT_e)\\ - (ne)^{-1}\vec{\nabla}\cdot[\vec{\vec{\Pi}}_e + nm\vec{v}\vec{v}]\\ - 0.71e^{-1}\nabla_{\|}T_e + \eta(\vec{j} - \vec{j}_T)\,,\end{aligned} \tag{3}$$

where

$$\vec{j} \equiv \vec{\nabla}\times\vec{B}\,, \tag{4}$$

$$\vec{j}_T \equiv (3/2)n\vec{B}\times\vec{\nabla}T_e/B^2. \tag{5}$$

Here S is a particle source, $\vec{E}_0$ is an external toroidal electric field, and m is the electron mass. The vector $\vec{u}$ is the ion flow and $\vec{v}$ is the electron flow, where $\vec{v} = \vec{u} - \vec{j}/(ne)$. Equation (2) is the quasistatic one-fluid force balance equation, wherein the viscous stresses have been neglected as they are small. Equation (3) is the electron force balance equation wherein the vector $\vec{j}_T$ is the cross-field thermoelectric force of Braginskii and the viscous and inertial stresses have been retained. In what follows, we will assume that the equilibrium is toroidally axisymmetric and that the $\vec{B}$ field is strictly poloidal and given by

$$\vec{B} = \vec{\nabla}\phi\times\vec{\nabla}\psi\,, \tag{6}$$

where (R,ϕ,z) are cylindrical coordinates and ψ is the flux function. The above system has to be augmented by equations for T_e and T_i which we will introduce soon. For immediate purposes, we will need only the fact that the parallel thermal conduction is very large and thus $T_e = T_e(\psi)$ and $T_i = T_i(\psi)$. (The latter presumes that heat sources are axisymmetric. For simplicity, we assume this to be the case for this paper. Nonaxisymmetric effects are discussed briefly at the end.)

It follows immediately, from the component of (2) along $\vec{B}$, that $n = n(\psi)$. Further, from (2), $\vec{j}_\perp = \vec{B}\times\vec{\nabla}(nT_e + nT_i)/B^2$ and we find $\vec{\nabla}\cdot\vec{j}_\perp = 0$ (for $B_\phi = 0$ and axisymmetry). In that case, $j_\|$ can be set to zero. Now, from the $\vec{\nabla}\phi$ component of (3), we get

$$\begin{aligned}-RE_0 = \vec{u}\cdot\vec{\nabla}\psi + (ne)^{-1}\vec{\nabla}\cdot[(\vec{\vec{\Pi}}_e + nm\vec{v}\vec{v})\cdot R^2\nabla\phi]\\ + \eta[(nT_e)' + (nT_i)' - (3/2)nT_e']R^2.\end{aligned} \tag{7}$$

The viscous and inertial terms in (7) can be shown to be negligible, as follows. First, consider the parallel viscous stress, the largest of the anisotropic stresses as formulated by Braginskii. This stress has the form $\vec{\vec{\Pi}} \sim (\hat{b}\hat{b} - \vec{\vec{1}}/3)W$, where W is a scalar. In that case, the divergence of the $\nabla\phi$ component of this stress vanishes identically given the symmetries of our problem. Next, we examine the collisionless, gyroviscous stress and the collisional, cross-field stress. We find these stresses to be smaller than the resistive term in Eq. (7) and, so, negligible. To see this, we note that the resistive term in (7) can be written $\eta R^2 j\cdot\nabla\phi$, where $\eta = m\nu_e/(ne^2)$ (here, ν_e is the electron collision frequency, ρ_e is the electron gyroradius, and v_e is the electron thermal speed). The viscous stresses, $\vec{\vec{\Pi}}$, are of order $nm\mu\nabla\nabla\vec{v}$, where $|\vec{v}|$ is of order the diamagnetic speed and μ is of order the Bohm diffusion coefficient, $\rho_e v_e$, or the cross-field diffusion, $\rho_e^2\nu_e$. With these sizes, the resistive term can be shown to be larger than the collisionless and the collisional stresses, respectively, by factors of at least $|\rho_e\nabla|^{-1}$ or $|\rho_e\nabla|^{-2}$. Finally, the inertial term can be estimated to be at most of order the collisionless stress. Hence, all the viscous and the inertial terms can be neglected. The resulting equation can be written

$$-RE_0 = \vec{u}\cdot\vec{\nabla}\psi + \eta[(nT_e)' + (nT_i)' - (3/2)nT_e']R^2. \tag{8}$$

We now take flux surface averages of (1) and the heat equations. Noting that $\langle\vec{\nabla}\cdot\vec{A}\rangle \equiv (d/d\psi)\langle\vec{\nabla}\psi\cdot\vec{A}\rangle$, where $\langle f\rangle \equiv \oint(d\ell/B)f$, the average of (1) yields

$$\frac{d}{d\psi}[n\langle\vec{u}\cdot\vec{\nabla}\psi\rangle] = \langle S\rangle\,. \tag{9}$$

For the heat equations, we use Braginskii's heat transport equations. The flux surface averages yield

$$\frac{3}{2}n\frac{dT_i}{d\psi}\langle\vec{u}\cdot\vec{\nabla}\psi\rangle + nT_i\frac{d}{d\psi}\langle\vec{u}\cdot\vec{\nabla}\psi\rangle = \frac{d}{d\psi}\left[\langle\kappa_{\perp i}R^2B^2\rangle\frac{dT_i}{d\psi}\right] - \nu_e\frac{m}{M}\langle n\rangle(T_i - T_e) + \langle H_i\rangle\,, \tag{10}$$

$$\begin{aligned}\frac{3}{2}n\frac{dT_e}{d\psi}\langle\vec{u}\cdot\vec{\nabla}\psi\rangle + nT_e\frac{d}{d\psi}\langle\vec{u}\cdot\vec{\nabla}\psi\rangle = \frac{d}{d\psi}\left\{\langle\kappa_{\perp e}R^2B^2/n\rangle\left[n\frac{dT_e}{d\psi} - \frac{9}{28}\frac{d}{d\psi}(nT_e + nT_i)\right]\right\}\\ - \nu_e\frac{m}{M}\langle n\rangle(T_e - T_i) + \langle\eta(\vec{j} - \vec{j}_T)\cdot\vec{j}\rangle + \langle H_e\rangle\,,\end{aligned} \tag{11}$$

where $\kappa_{\perp\alpha}$ are the perpendicular thermal conductivities [2]. In the above heat equations, all the Braginskii terms except for the viscous terms have been accounted for (the viscous terms can be estimated to be unimportant). In particular, the drift and thermal heat fluxes as well as Ohmic heating have been included: "Pfirsch-Schluter" effects do not arise in this system since there is no toroidal magnetic field.

Finally, we take the $\vec{\nabla}\psi$ component of (2) to obtain the Grad-Shafranov equation

$$\vec{\nabla} \cdot (R^{-2}\vec{\nabla}\psi) = -(nT_e + nT_i)'. \tag{12}$$

Equations (8)–(12) constitute a closed set of equations for $n(\psi)$, $T_\alpha(\psi)$, ψ, and $\langle \vec{u} \cdot \vec{\nabla}\psi \rangle$. Particle sources, $\langle S \rangle$, drive $\langle \vec{u} \cdot \vec{\nabla}\psi \rangle$ [from (9)], and heat sources, $\langle H \rangle$, determine $T(\psi)$ profiles [from (10) and (11)]. Given $\langle \vec{u} \cdot \vec{\nabla}\psi \rangle$ and T_α, the $n(\psi)$ profile can be deduced from (8) if E_0 is provided. Finally, n and T_α profiles determine ψ, from (12).

We now address the question of interest; namely, can an FRC be maintained in steady state without external transformers driving E_0. The answer is in the affirmative. In the simplest possible configuration, we set $E_0 = 0$ and $S = 0$. In this case, $\langle \vec{u} \cdot \vec{\nabla}\psi \rangle = 0$, from (9). It follows, from (8), that the current is driven completely by the thermoelectric force, i.e., $\vec{j} \rightarrow \vec{j}_T$. The following equilibrium configuration is obtained:

$$\frac{d}{d\psi}\left[\langle \kappa_{\perp i} R^2 B^2 \rangle \frac{dT_i}{d\psi}\right] = \nu_e \frac{m}{M} \langle n \rangle (T_i - T_e) - \langle H_i \rangle, \tag{13}$$

$$\frac{d}{d\psi}\left[\frac{29}{56} \langle \kappa_{\perp e} R^2 B^2 \rangle \frac{dT_e}{d\psi}\right] = -\nu_e \frac{m}{M} \langle n \rangle (T_i - T_e) - \langle H_e \rangle, \tag{14}$$

$$\frac{n'}{n} = \frac{(1/2)T_e' - T_i'}{T_e + T_i}, \tag{15}$$

$$\vec{\nabla} \cdot (R^{-2}\vec{\nabla}\psi) = (nT_e + nT_i)'. \tag{16}$$

Thus, the heat sources drive the entire system and maintain a steady state: the temperatures T_α are determined from (13) and (14); this yields n from (15), which then determines ψ from (16). From (15), we note that $n \propto T_e^{1/2}$ for $T_e \gg T_i$, and $n \propto T_e^{-1/4}$ for $T_e = T_i$. Thus, n can be hollow (n is flat for $T_e = 2T_i$) but note that the pressure is always peaked for peaked T_e, since $(nT_e + nT_i)' = (3/2)nT_e'$, from (8).

We have solved the time-dependent version of the system (1)–(5) by numerical means. A simple one-dimensional system was investigated; i.e., the magnetic field was unidirectional but reversing sign at the axis. The ion temperature was set to zero and constant transport coefficients were used, for simplicity. Three questions were posed and resolved: (1) A steady state is achieved by simply heating the axis, as in Eqs. (13)–(16); (2) such a state is stable; and (3) the magnetic field can be "pumped up" by heating on axis if it is initially very weak (i.e., the initial pressure profile is almost flat). We found that a steady state is indeed achieved, that such a state is stable, and that the magnetic field can, in fact, be pumped up to the desired value.

We point out here that the thermoelectric force introduces considerations that obviate Cowling's theorem [7]. Let us state this theorem: in resistive MHD, where Ohm's Law can be written $\vec{E} = -\vec{u} \times \vec{B} + \eta\vec{j}$, if the system is axisymmetric, the poloidal component of the magnetic field must decay due to resistivity. Now Eq. (3) is just the generalized Ohm's Law, including the above and more. Equation (8) is the 2D version of one of the components of (3), wherein the first two terms on the right-hand side constitute the usual ηj term [using Eq. (12)] and the third term is the thermoelectric term. The time dependence in Eq. (8) can be restored by adding $-\partial\psi/\partial t$ to the left-hand side. Now consider (8), for $E_0 = 0$, as one approaches the magnetic axis (i.e., the limit of $\vec{\nabla}\psi \rightarrow 0$). In this case, if $|\vec{u}|$ is finite at the axis, we have $\partial\psi/\partial t \rightarrow \eta\nabla^2\psi + (3/2)\eta R^2 n T_e'$. If the thermoelectric term were absent, ψ must decay—this is Cowling's theorem. Clearly, this tendency is overcome by the thermoelectric term. The thermoelectric term constitutes a source for the creation of magnetic flux; in this sense, even if this source term were zero near the magnetic axis (as we elaborate below), a steady state solution for ψ still exists since flux annihilation at the axis is balanced by flux creation elsewhere. In particular, if heat sources are present to maintain temperature gradients, then ψ cannot only be maintained but also be pumped up, rendering a 2D dynamo effect. (Note: $|T_e'| = |\vec{\nabla}T_e/RB|$; thus, near the axis, $|\vec{\nabla}T_e| \rightarrow 0$ and $B \rightarrow 0$, but T_e' is finite [3].)

We may elaborate on this discussion of Cowling's theorem as follows. Suppose we have a system where we have no inductive electric field, E_0, but we have a source of particles, S, and a heat source, H, both, say, feeding the O point. For simplicity, let us examine the case where R is large and the flux surfaces are circular so that $(d/d\psi) \rightarrow (1/B)(d/dr)$. Consider first the case where the thermoelectric effect is absent. We look for a steady state solution with nonzero S and H with the idea being that if the pressure gradients can be maintained by the particle and heat input, this will maintain the diamagnetic currents and thus the whole magnetic configuration. From Eq. (8), we get $d(rB)/dr = (u_r/\eta)(rB)$, where we have used (12). The radial flow, u_r, is specified by S, according to Eq. (9) which becomes $nru_r = \int_0^r dr'\, r'S(r')$. In this case, we may integrate the previous equation to solve for rB, viz., $rB = C\exp[\int^r dr'\, u_r/\eta]$, where C is the integration constant. Upon applying the boundary condition $B(0) = 0$, we find $C = 0$. Thus, the only steady state solution is for vanishing B, i.e., the magnetic field diffuses away to zero in spite of particle and heat input. This is another way of proving Cowling's theorem, in the presence of sources. Let us now restore the thermoelectric term. Let the particle source be zero, for simplicity. The corresponding steady state equation, obtained from (8) and (12), is now $(rB)\, d(rB)/dr = -(3/2)nr^2 f(r)\, dT_e/dr$, where we have introduced a new function $f(r)$ which is arbitrary for the moment but will be discussed soon. From this equation, we find the steady state solution for $B(r)$: $B^2(r) = -(3/r^2)\int_0^r dr'\, nr'^2 f(r')\, dT_e/dr'$. This solution is well behaved and satisfies the boundary conditions. Thus, the thermoelectric force can give rise to a steady state solution, obviating Cowling's theorem. The function $f(r)$ has been inserted to show that this solution is good even if the thermoelectric effect were to go to zero at the origin; i.e., a steady state solution for $B(r)$ exists even if $f(r) \to 0$ as $r \to 0$. We discuss this point further below.

Can a scheme for maintaining FRC's in steady state by central heating alone be feasible for fusion? Insofar as the system is steady state and, so, to be maintained for many energy confinement times, it is clear that the FRC must be heated by auxiliary means, such as neutral beam heating (NBI), for example. The above analysis shows, pleasingly, that the same auxiliary heat source will automatically be sufficient to maintain the magnetic field. In other words, if FRC's can be heated, the magnetic field will also be maintained: in the transient phase the pressure profiles will adjust, but the system will settle down to the density profile given by (15) in steady state (if there is also a particle source, as with NBI, there will be a cross-field flow that will alter the balance above, though this will still not affect the steady state aspect).

One major concern is that if the transport is anomalous, then the "detailed balance," signified by the cancellation of $\eta\vec{j}$ with $\eta\vec{j}_T$ [cf. Eq. (3)], could be thrown off depending on how turbulence affects the cross-field resistivity and/or thermoelectric effects. Another issue is the validity of conclusions drawn from Chapman-Enskog theory [2] as they are applied to collisionless thermonuclear grade plasma. A detailed assessment of the latter is made in Ref. [3]. Briefly, the thermoelectric force can be expected to survive into long mean free path regimes since this force is a cross-field phenomenon. In addition, and more to the point, the expected failure of the small Larmor radius approximation near the O point is not serious since the thermoelectric effect has to do with electron gyroradius effects and electrons become unmagnetized only in a very small region near the O point (less than 1 cm for typical fusion parameters). The latter point is of some importance in reference to Cowling's theorem. For, near the axis as $B \to 0$, the thermoelectric effect could vanish [2,8] (all diamagnetic effects vanish). However, as we have discussed above, this is not an issue since the thermoelectric source of flux is nonzero almost everywhere (for $r > 1$ cm in the plasma) and will actually *create* flux there. In fact, as we have shown in the preceding paragraphs, a steady state solution for $B(r)$ can be obtained with the thermoelectric force even if the latter were to vanish as $r \to 0$ [see the solution for $B(r)$ above and the discussion on the function $f(r)$].

It is clear, though, that only experiment can resolve all the above issues. The proposed SPIRIT (self-organized plasma with inductive, reconnection, and injection techniques) experiment [9] is in a position to explore these questions. The experiment will form an FRC with a concomitant toroidal NBI. The NBI will provide the necessary auxiliary heat but, if the above physics works, and if the plasma is MHD stable, it should also provide a thermoelectric current drive sufficient to prevent flux decay. A simpler or an existing experiment that tests just the basic nature of the thermoelectric physics is desirable. The levitated dipole experiment [10], currently under construction, may serve this purpose well. It should be noted that in both these experiments, the heat sources will be nonaxisymmetric. Since there is no toroidal field in the system, the cross-field heat conduction is relatively weak and, as a result, one can expect toroidal asymmetries in the temperature. In our analysis in this paper, we have assumed that the heat sources are axisymmetric; thus, at a minimum, the analysis should be redone allowing nonaxisymmetric heating. It is well known [10] that such asymmetries lead to convection cells with toroidal structure. A transport analysis with nonaxisymmetric heating and convection is tractable in simple cases: we have examined the case of a Z pinch. We find that it is only the toroidally averaged radial flow, not the toroidally varying convection, that has any influence on flux transport. Insofar as the averaged radial flow results from particle sources and does not unduly influence the thermoelectric effect (as we have shown herein), it is possible that the asymmetries may not affect the thermoelectric effect we have investigated herein. We may also add that if large toroidal flow is driven in an FRC, as is proposed for the SPIRIT experiment, this may have the effect of smoothing out the toroidal asymmetries.

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