

Unified Treatment of Small- and Large-Scale Dynamos in Helical Turbulence

Kandaswamy Subramanian

National Centre for Radio Astrophysics, TIFR, Poona University Campus, Ganeshkhind, Pune 411 007, India

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Helical turbulence may provide the key to the generation of large-scale magnetic fields. Turbulence also generically leads to rapidly growing small-scale magnetic fields. These two processes are usually studied separately. We give here a unified treatment of both processes, in the case of random fields, incorporating also a simple model nonlinear drift. We uncover an interesting plausible saturated state of the small-scale dynamo and a novel analogy between quantum mechanical (QM) tunneling and the generation of large-scale fields. A field generated by the small-scale dynamo, can “tunnel” to produce large-scale correlations, which in steady state, correspond to a force-free “mean” field.

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Large-scale magnetic fields in astronomical objects are thought to be generated by dynamo action involving helical turbulence and rotational shear [1,2]. Here large-scale refers to scales much larger than the outer scale, say L , of the turbulence. However, turbulent motions, with a large enough magnetic Reynolds number (MRN), can also excite a small-scale dynamo, which exponentiates fields correlated on the turbulent eddy scale, at a rate much faster than the mean field growth rate [1,3]. These two dynamo problems, viz. the small-scale dynamo (SSD) and large-scale dynamo (LSD), are usually treated separately. However, this separation is often artificial; there is no abrupt transition from the field correlated on scales smaller than L and that correlated on larger scales. We show here that the equations for the magnetic correlations, which involve both the longitudinal and helical parts, are already sufficiently general to incorporate both small- and large-scale dynamos in the case of random fields. They provide us with a paradigm to study the dynamics in a unified fashion, which could be particularly useful to study a possible inverse cascade of magnetic fields, to scales larger than L .

Consider the induction equation for the magnetic field,

$$(\partial \mathbf{B} / \partial t) = \nabla \times (\mathbf{v} \times \mathbf{B} - \eta \nabla \times \mathbf{B}), \quad (1)$$

where \mathbf{B} is the magnetic field, \mathbf{v} is the velocity of the fluid, and η is the Ohmic resistivity. Take $\mathbf{v} = \mathbf{v}_T + \mathbf{v}_D$, the sum of an externally prescribed stochastic field \mathbf{v}_T , and a drift component \mathbf{v}_D , which models the nonlinear back reaction of the growing Lorentz force. We assume \mathbf{v}_T to be an isotropic, homogeneous, Gaussian random velocity field with zero mean. For simplicity, we also assume \mathbf{v}_T to have a delta function correlation in time (Markovian approximation). Its two point correlation is specified as $\langle v_T^i(\mathbf{x}, t) v_T^j(\mathbf{y}, s) \rangle = T^{ij}(r) \delta(t - s)$ with (cf. [4])

$$T^{ij}(r) = T_N \left[\delta^{ij} - \left(\frac{r^i r^j}{r^2} \right) \right] + T_L \left(\frac{r^i r^j}{r^2} \right) + C \epsilon_{ijf} r^f. \quad (2)$$

Here $\langle \rangle$ denotes ensemble averaging, $r = |\mathbf{x} - \mathbf{y}|$, and $r^i = x^i - y^i$. $T_L(r)$ and $T_N(r)$ are the longitudinal and transverse correlation functions for the velocity field while $C(r)$ represents the helical part of the velocity correlations. If $\nabla \cdot \mathbf{v}_T = 0$, $T_N = (1/2r) d(r^2 T_L) / dr$. (Note \mathbf{v}_T is also not correlated with the magnetic field.)

To model the drift velocity in a tractable manner, but one which nevertheless gives some feel for possible nonlinear effects, we proceed as follows. As the magnetic field grows, the Lorentz force pushes on the fluid. We assume the fluid almost instantaneously responds to this push and develops an extra “drift” component to the velocity proportional to the instantaneous Lorentz force. So we take a model $\mathbf{v}_D = a[(\nabla \times \mathbf{B}) \times \mathbf{B}]$, with the parameter $a = \tau / 4\pi\rho$, where τ is some response time, and ρ is the fluid density. (Such a velocity can also arise when friction dominates inertial forces on ions, as in ambipolar drift.) This gives a model nonlinear problem, where the nonlinear effects of the Lorentz force are taken into account as a simple modification of the velocity field. Such a phenomenological modification of the velocity field has in fact been used by Pouquet *et al.* [5] and Zeldovich *et al.* [1] (p. 183) to discuss nonlinear modifications to the alpha effect, and we adopt it below.

Consider a system, whose size $S \gg L$, and for which the mean field averaged over any scale is zero. Of course, the concept of a large-scale field still makes sense, as the correlations between field components separated at scales $r \gg L$ can, in principle, be nonzero. We take \mathbf{B} to be a homogeneous, isotropic, Gaussian random field with zero mean. This is a perfectly valid assumption to make in the kinematic regime ($\mathbf{v}_D = 0$), as the stochastic \mathbf{v}_T has these symmetries. When we include the nonlinear drift velocity, it amounts to making a closure hypothesis, which we do here again for analytical tractability. The equal-time, two point correlation of the magnetic field is given by $\langle B^i(\mathbf{x}, t) B^j(\mathbf{y}, t) \rangle = M^{ij}(r, t)$, where

$$M^{ij} = M_N \left[\delta^{ij} - \left(\frac{r^i r^j}{r^2} \right) \right] + M_L \left(\frac{r^i r^j}{r^2} \right) + H \epsilon_{ijf} r^f. \quad (3)$$

(Here $\langle \rangle$ denotes a double ensemble average over both the stochastic velocity and stochastic \mathbf{B} fields.) $M_L(r, t)$ and $M_N(r, t)$ are the longitudinal and transverse correlation functions for the magnetic field while $H(r, t)$ represents the (current) helical part of the correlations. Since $\nabla \cdot \mathbf{B} = 0$, $M_N = (1/2r) \partial(r^2 M_L) / (\partial r)$.

The stochastic Eq. (1) can be converted into the evolution equations for M_L and H . We give a detailed derivation of these equations elsewhere, including the effect of the nonlinear drift [6]. We get

$$\frac{\partial M_L}{\partial t} = \frac{2}{r^4} \frac{\partial}{\partial r} \left(r^4 \kappa_N \frac{\partial M_L}{\partial r} \right) + G M_L - 4 \alpha_N H, \quad (4)$$

$$\frac{\partial H}{\partial t} = \frac{1}{r^4} \frac{\partial}{\partial r} \left(r^4 \frac{\partial}{\partial r} (2 \kappa_N H + \alpha_N M_L) \right), \quad (5)$$

where we have defined

$$\begin{aligned} \kappa_N &= \eta + T_L(0) - T_L(r) + 2aM_L(0, t), \\ \alpha_N &= 2C(0) - 2C(r) - 4aH(0, t), \\ G &= -4[(T_N/r)' + (rT_L)/r^2]. \end{aligned} \quad (6)$$

Here prime denotes a derivative with respect to r . These equations form a closed set of nonlinear partial differential equations for the evolution of M_L and H , describing, as we will see, both SSD and LSD action for random fields. The effective diffusion κ_N includes microscopic diffusion (η), a scale-dependent turbulent diffusion [$T_L(0) - T_L(r)$], and nonlinear drift adds an amount $2aM_L(0, t)$, proportional to the energy density in the fluctuating fields. Similarly α_N represents a scale-dependent α effect [$2C(0) - 2C(r)$] and nonlinear drift decreases this by $4aH(0, t)$, proportional to the mean current helicity of the magnetic fluctuations. This modification to the α effect is the same as that obtained in [1,5]. The $G(r)$ term allows for the rapid generation of magnetic fluctuations by velocity shear and the existence of a SSD independent of any large-scale field [1,3].

First consider nonhelical turbulence, with $C(r) = 0$, allowing solutions with $H(r, t) = 0$. This case has been extensively studied for the kinematic case ($a = 0$), when $T_L(r)$ has a single scale (cf. [1,3]) and by us for a model Kolmogorov-type turbulence [6]. These studies show that one has SSD action, and magnetic fields correlated on scales up to the turbulent scale can be generated. In the kinematic limit, κ_N and G are time independent. One can then look for eigenmode solutions to (4), of the form $\Psi(r) \exp(2\Gamma t) = r^2 \sqrt{\kappa_N} M_L$. This transforms Eq. (4) for $M_L(r, t)$ into a time independent, Schrödinger-type equation, but with a variable (and positive) mass,

$$-\Gamma \Psi = -\kappa_N \frac{d^2 \Psi}{dr^2} + U_0(r) \Psi. \quad (7)$$

The “potential” is $U_0(r) = T_L'' + (2T_L'/r) + \kappa_N''/2 - (\kappa_N')^2/(4\kappa_N) + 2\kappa_N/r^2$, for a divergence-free ve-

locity field. The boundary condition is $\Psi \rightarrow 0$, as $r \rightarrow 0, \infty$. Note that $U_0 \rightarrow 2\eta/r^2$ as $r \rightarrow 0$, while $U_0 \rightarrow 2[\eta + T_L(0)]/r^2$ as $r \rightarrow \infty$. The possibility of growing modes with $\Gamma > 0$ obtains, if one can have a potential well, with U_0 sufficiently negative in some range of r , to allow the existence of *bound states*, with an “energy” $E = -\Gamma < 0$.

Suppose we have turbulent motions on a single scale L , with a velocity scale v . Define the magnetic Reynolds number $R_m = vL/\eta$. Then one finds [3,6] that there is a critical MRN, $R_m = R_c \approx 60$, so that for $R_m > R_c$, the potential U_0 allows the existence of bound states. For $R_m = R_c$, $\Gamma = 0$, and this marginal stationary state is the “zero” energy eigenstate in the potential U_0 [7]. For $R_m > R_c$, $\Gamma > 0$ modes of the SSD can be excited, and the fluctuating field correlated on a scale L grows exponentially on the corresponding “eddy” turnover time scale, with a growth rate $\Gamma_L \sim v/L$ [8]. To understand the spatial structure of the fields, define $w(r, t) = \langle \mathbf{B}(\mathbf{x}, t) \cdot \mathbf{B}(\mathbf{y}, t) \rangle = (1/r^2) d[r^3 M_L]/dr$, the correlated dot product of the random field. For the SSD, $w(r)$ is strongly peaked within a region $r = r_d \approx L(R_m)^{-1/2}$ about the origin for all the modes, and for the fastest growing mode, changes sign across $r \sim L$ and rapidly decays with increasing r/L . (For the marginal mode with $\Gamma = 0$, R_m is replaced by R_c .) Note that r_d is the diffusive scale satisfying the condition $\eta/r_d^2 \sim v/L$. A pictorial interpretation of the correlation function, due to the Zeldovich school (cf. [1,3]), is to think of the field as being concentrated in “flux ropes” with thickness of order $r_d \ll L$, and curved on a scale up to $\sim L$, to account for negative values of w .

How does the SSD saturate? The back reaction, in the form of a nonlinear drift, simply replaces η by an effective, time dependent $\eta_D = \eta + 2aM_L(0, t)$, in the κ_N term of Eq. (4). Suppose we define an effective MRN, for fluid motion on scale L , by $R_D(t) = vL/\eta_D(t)$. Then as the energy density in the fluctuating field, say $E_B(t) = 3M_L(0, t)/8\pi$, increases, R_D decreases. In the final saturated state, with $(\partial M_L/\partial t) = 0$ (obtaining say at time t_s), M_L , and hence the effective η_D in (4) become independent of time. Solving for this stationary state then becomes identical to solving for the marginal (stationary) mode of the kinematic problem, except that R_m is replaced by $R_D(t_s)$. The final saturated state is then the marginal eigenmode which obtains, when E_B has grown (and R_D decreased) such that $R_D(t_s) = vL/[\eta + 2aM_L(0, t_s)] = R_c \sim 60$. Also $w(r)$ for the saturated state will be strongly peaked within a region $r = r_d \approx L(R_c)^{-1/2}$ about the origin, change sign across $r \sim L$, and then rapidly decay for larger r/L . From the above constraint, $M_L(0, t_s) = vL/(2aR_c)$, assuming $\eta \ll 2aM_L(0, t_s)$. So at saturation,

$$E_B(t_s) = \frac{3M_L(0, t_s)}{8\pi} = \frac{3}{2} \frac{\rho v^2}{2} \frac{L/v}{\tau} \frac{1}{R_c}. \quad (8)$$

Note that τ is an unknown model parameter. If we were to adopt $\tau \sim L/v$, that is the eddy turnover time, then E_B

at saturation is a small fraction $\sim R_C^{-1} \ll 1$ of the equipartition value. Further suppose we interpret $w(r)$ for the saturated state in terms of the Zeldovich *et al.* picture of flux ropes of thickness r_d , curved on scale L , in which a field of strength B_p is concentrated. In this picture, the average energy density in the field $E_B \sim (B_p^2/8\pi)Lr_d^2/L^3$. Using $r_d^2/L^2 \approx R_c^{-1}$, and $\tau \sim L/v$, we then have $B_p^2/8\pi \sim \rho v^2/2$, where, remarkably, the R_c^{-1} dependence has disappeared, and B_p has equipartition value. So the SSD could saturate with the small-scale field having peak values of order the equipartition field, being concentrated into flux ropes of thickness $LR_c^{-1/2}$, curved on scale L , and an average energy density R_c^{-1} times smaller than equipartition.

We now turn to consider the effect of helical correlations. If $\alpha_0 = 2C(0) \neq 0$, then one can see from Eqs. (4) and (5) that new generation terms arise at $r \gg L$, due to the α effect, in the form $\dot{M}_L = \dots - 4\alpha_T H$ and $\dot{H} = \dots + [r^4(\alpha_T M_L)]/r^4$. Here $\alpha_T = \alpha_0 - 4aH(0, t)$ and the dot represents a time derivative. These couple M_L and H and lead to the growth of large-scale correlations. There is also a decay of the correlations at $r \gg L$, due to diffusion with an effective diffusion coefficient, $\eta_T = \eta + T_L(0) + 2aM_L(0, t)$. From dimensional analysis, the effective growth rate is $\Gamma_D \sim \alpha_T/D - \eta_T/D^2$, for correlations on scale $\sim D$, as in the large-scale α^2 dynamo. This also picks out a special scale $D_0 \sim \eta_T/\alpha_T$ for a stationary state (see below). Further, as the SSD is simultaneously leading to a growth of M_L at $r < L$, in general at a faster rate $v/L \gg \alpha_0/D$, the growth of large-scale correlations can be seeded by the tail of the SSD eigenfunction at $r > L$. The SSD generated small-scale field can thus seed the large-scale dynamo. Indeed, as advertised, both the SSD and LSD operate simultaneously when $\alpha_0 \neq 0$ and can be studied simply by solving for one $M_L(r, t)$.

The coupled time evolution of H and M_L for a nonzero α_0 requires numerical solution. But interesting analytical insight into the system can be obtained for the marginal, stationary mode, with $(\partial M_L/\partial t) = (\partial H/\partial t) = 0$. In fact, both the kinematic and nonlinear dynamo problems can be treated in a unified fashion. With H independent of time, Eq. (5) implies $2\kappa_N H + \alpha_N M_L = 0$, for any solution regular at $r = 0$ and vanishing at $r \rightarrow \infty$. From this, as $r \rightarrow 0$, $2\eta H(0, t) = 0$, and hence $H(0, t) = 0$ for a nonzero η , a result which also follows directly from the evolution (conservation) of magnetic helicity [9]. So any general nonlinear addition to the α effect which arises in terms of $H(0, t)$ has to vanish in a stationary state.

Now substitute $H(r) = -\alpha_N(r)M_L(r)/[2\kappa_N(r)]$ into (4) and define once again $\Psi = r^2\sqrt{\kappa_N}M_L$. We get

$$-\kappa_N \frac{d^2\Psi}{dr^2} + \Psi \left[U_0 - \frac{4[C(0) - C(r)]^2}{\kappa_N} \right] = 0. \quad (9)$$

We see that the problem of determining the magnetic field correlations, for the marginal/stationary mode once again becomes the problem of determining the zero-energy

eigenstate in a modified potential, $U = U_0 - 4[C(0) - C(r)]^2/\kappa_N$. Note that the addition to U_0 , due to the helical correlations, is always negative definite. So helical correlations tend to make bound states easier to obtain. When $C(0) = 0$, and there is no net α effect, the addition to U_0 vanishes at $r \gg L$, and $U \rightarrow 2\eta_T/r^2$ at large r , as before. The critical MRN for the stationary state will, however, be smaller than when $C(r) \equiv 0$, because of the negative definite addition to U_0 .

When $\alpha_0 = 2C(0) \neq 0$, a remarkable change occurs in the potential. At $r \gg L$, where the turbulence velocity correlations vanish, we have $U(r) = 2\eta_T/r^2 - \alpha_0^2/\eta_T$. So the potential U tends to a negative definite constant value of $-\alpha_0^2/\eta_T$ at large r (and the effective mass, $1/2\kappa_N \rightarrow 1/2\eta_T$, a constant with r). There are strictly no bound states, with zero energy/growth rate, for which the correlations vanish at infinity. We have schematically illustrated the resulting potential U in Fig. 1, which is a modification of Fig. 8.4 of Zeldovich *et al.* [1]. In fact, for a nonzero α_0 , U corresponds to a potential which allows tunneling (of the bound state) in the corresponding quantum mechanical (QM) problem. It implies that the correlations are necessarily nonzero at large $r > L$. The analytical solution to (9) at large $r \gg L$ is easily obtained. We have for $r \gg L$, $M_L(r) = \bar{M}_L(r)$,

$$\bar{M}_L(r) = \frac{1}{r^{3/2}} [C_1 J_{3/2}(\mu r) + C_2 J_{-3/2}(\mu r)], \quad (10)$$

where $\mu = \alpha_0/\eta_T = D_0^{-1}$, and C_1, C_2 arbitrary constants. Also $w(r) = \bar{w}(r) = \mu r^{-1} [C_1 \sin \mu r + C_2 \cos \mu r]$. Clearly for a nonzero α_0 , the correlations in steady state at large r are like “free-particle” states, extending to infinity.

An alternate derivation of $\bar{M}_L(r)$, for the kinematic case, clarifies its meaning further. Suppose one thinks of the large-scale field as a “mean” field \mathbf{B}_0 , the mean taken over cells much larger than L . Assume \mathbf{B}_0 itself

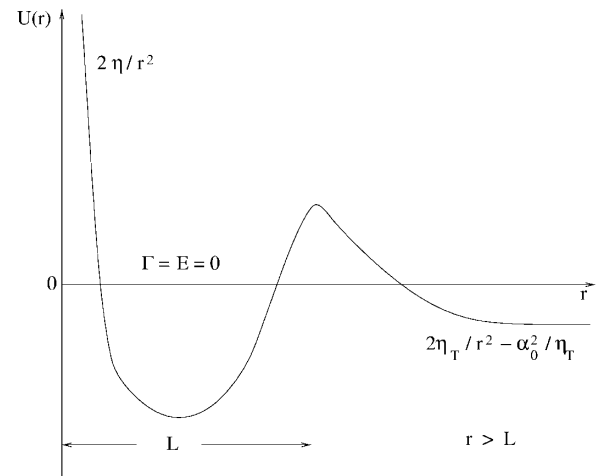


FIG. 1. Schematic illustration of the potential $U(r)$ for the marginal mode in helical turbulence. A nonzero α_0 allows the tunneling of the zero-energy state to produce large-scale correlations.

is random over different cells, statistically homogeneous and isotropic, with a correlation $\langle B_{0i} B_{0j} \rangle = M_{ij}^L(r)$. Let M_L^L , M_N^L , and H^L be, respectively, the corresponding longitudinal, transverse, and helical correlations. Then \mathbf{B}_0 in each cell obeys the kinematic, mean field dynamo equation $(\partial \mathbf{B}_0 / \partial t) = \nabla \times \{\alpha_0 \mathbf{B}_0 - [\eta + T_L(0)] \nabla \times \mathbf{B}_0\}$, whose steady state solution is $\nabla \times \mathbf{B}_0 = \mu_0 \mathbf{B}_0$, where $\mu_0 = \mu(a = 0)$. This constraint, imposed on M_{ij}^L , gives $H^L = -\mu_0 M_L^L/2$ and $(M_L^L - M_N^L)/r^2 - (M_N^L)' / r = \mu_0^2 M_L^L/2$. Also as $\nabla \cdot \mathbf{B}_0 = 0$, $M_N^L = (1/2r)(r^2 M_L^L)'$. These three equations fix all the functions uniquely. We get, remarkably, $M_L^L(r) = \tilde{M}_L(r)$, with μ replaced by the kinematic value μ_0 . So this solution actually describes a random mean field, for the marginal large-scale dynamo. Similarly if we had imposed $\nabla \times \mathbf{B}_0 = \mu \mathbf{B}_0$, M_L^L would be given by Eq. (10). Note this also shows that the effective, steady state, large-scale field \mathbf{B}_0 is force-free, although \mathbf{B} itself is not.

It is straightforward to connect the large-scale, force-free field for the marginal mode of helical turbulence, with the SSD generated field, as they are both the solution of the same Eq. (9), for large and small r , respectively. For example, one can integrate Eq. (9), adopting different starting values of $M_L(0, t_s)$ and taking $M_L^L(0, t_s) = 0$ to construct a whole family of solutions [parametrized by $M_L(0, t_s)$], which match small-scale correlations with the large-scale correlation of (10). For each such solution, we will have one value of C_1/C_2 . Note that this is unlike the standard QM tunneling problem, where the boundary condition that the free-particle state is an outgoing wave at large r uniquely fixes the tunneling amplitude, for a given bound state. However, when we consider a zero-energy, stationary state, there is no such natural time-asymmetric boundary condition; so there is no unique fix for C_1/C_2 in (10). Nevertheless, if $M_L(0, t_s)$ is so small and R_D so large that U admits bound states with energy $E < -\alpha_0^2/\eta_T$, then the corresponding time dependent system is unlikely to lead to stationary correlations. This sets a lower bound on $M_L(0, t_s)$ or E_B . Further as $\dot{M}_L(0, t) = \dots + 16H^2(0, t)$, the saturation of $M_L(0, t)$ depends on how fast $H(0, t) \rightarrow 0$, its stationary value, over and above the saturation effects of the increasing diffusion η_D . So the full time dependent problem needs to be solved to fix an upper bound on $M_L(0, t_s)$ or E_B [10]. We will return to this elsewhere.

In conclusion, we have given here a unified treatment of small- and large-scale dynamos for the case of random fields, incorporating also a simple model nonlinear drift. We uncovered an interesting plausible saturated state of the small-scale dynamo. For random fields, we argued that any nonlinear addition to the α effect in terms of the average current helicity $H(0, t)$ (cf. [1,5]) has to vanish in

a steady state. The steady state problem of the combined small-/large-scale dynamo was then mapped to a zero-energy, QM potential problem; but a potential which, for $\alpha_0 \neq 0$, allows tunneling of bound states. A field generated by the SSD, can then “tunnel” to seed the growth of large-scale correlations, which in steady state, correspond to a force-free mean field. It remains to solve the fully time dependent system and to incorporate more realistic back-reaction effects of the Lorentz force.

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 - [7] As the “mass” $(1/2\kappa_N)$ is variable, one may also think in terms of an effective potential U_0/κ_N for the zero-energy bound state, without changing the qualitative remarks regarding bound states, and the possibility of tunneling when $\alpha_0 \neq 0$, mentioned later. We follow the convention of Zeldovich *et al.* in [1] in calling U_0 the potential.
 - [8] In Kolmogorov-type turbulence, the mode correlated on the cutoff scale grows fastest. But modes with extent up to the outer scale L can also grow at their corresponding, slower eddy turnover rates [3,6].
 - [9] From (4) and (5), $(dI_M/dt) = -2\eta \langle \mathbf{B} \cdot (\nabla \times \mathbf{B}) \rangle = 12\eta H(0, t)$, where $I_M = \langle \mathbf{A} \cdot \mathbf{B} \rangle$ is proportional to the magnetic helicity and \mathbf{A} is the vector potential. So as $\eta \rightarrow 0$, I_M is conserved in the ideal limit, as required [2]. But for stationary states with $dI_M/dt = 0$, when $\eta \neq 0$, then $H(0, t) = 0$.
 - [10] Solving the time dependent system will also clarify how initially localized correlations “propagate” to larger and larger scales, due to the α_T coupling, and asymptotically reach the large r stationary state [viz. Eq. (10)].