Boundary S Matrix and the Anti–de Sitter Space to Conformal Field Theory Dictionary

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An S matrix analog is defined for anti–de Sitter (Ads) space by constructing "in" and "out" states that asymptote to the timelike boundary. A derivation parallel to that of the Lehmann-Symanzik-Zimmermann formula shows that this "boundary S matrix" is given directly by correlation functions in the boundary conformal theory. This provides a key entry in the AdS to conformal field theory dictionary.

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The conjectured anti–de Sitter (AdS)/conformal field theory (CFT) correspondence [1] has offered a promising new window into the dynamics of string and M theory. But in order to exploit this powerful framework, we must decipher the holographic relationship between the bulk and boundary theories. Gubser *et al.* [2] and Witten [3] made important progress in this regard by providing a CFT to AdS dictionary: They show how to derive CFT correlation functions from the bulk theory in AdS. This has allowed the successful calculation of various CFT correlators.

However, in order to study bulk physics, and in particular to understand the undoubtedly profound implications of holography, a reverse dictionary is needed: We need to know which bulk quantities can be calculated, and how to calculate them, from the boundary CFT. Another important and closely related question is how to treat scattering in AdS. Because of the periodicity of particle orbits and lack of ordinary asymptotic states in AdS, a conventional S matrix cannot be defined (see [4] for more discussion). However, Ref. [4] outlined the definition of an AdS analog of the S matrix in terms of scattering of states from the timelike infinity. References [5,6] gave a related definition in the infinite-N limit. The purpose of this paper is to go further and provide an intrinsic and explicit definition of this "boundary S matrix" for arbitrary N, and to give a precise relation between it and the CFT correlators. The discussion also clarifies the relation between the framework of [5,6] and that of [4]. Other recent treatments of related aspects of the AdS/CFT dictionary include Refs. [7–9].

To summarize in advance, the boundary S matrix will be defined as an overlap of certain "in" and "out" states. These will be defined so that they correspond to particles asymptotic to the timelike boundary of AdS in the past and/or future. An AdS analog of the Lehmann-Symanzik-Zimmermann (LSZ) formula can then be derived and can relate this S matrix to the bulk correlation functions. Finally, the results of [3] are used to rewrite the boundary S matrix in terms of the CFT correlation functions. An extremely simple relationship results: The boundary S matrix *equals* the corresponding CFT correlator. This serves as a key entry in the AdS to CFT dictionary.

For simplicity we will consider scalar fields, with action

$$S = -\int dV \left[\frac{1}{2} (\nabla\Phi)^2 + \frac{m^2}{2} \Phi^2 + U(\Phi) \right], \qquad (1)$$

where U summarizes the interaction terms. The generalization to other fields should be straightforward. We will work in global coordinates $x = (t, \rho, \Omega)$ for AdS_{d+1},

$$ds^2 = R^2(-\sec^2\rho\, dt^2 + \sec^2\rho\, d\rho^2 + \tan^2\rho\, d\Omega_{d-1}^2), \qquad (2)$$

although translation to Poincaré coordinates should also be straightforward.

Certain facts about the solutions to the free equations will be useful in the following. The effective gravitational potential of anti–de Sitter space confines particles to its interior. Solutions to the free equation

$$(\Box - m^2)\phi = 0 \qquad (3)$$

therefore exist at arbitrary frequency ω, but are only normalizable (in the Klein-Gordon norm) for a discrete set of frequencies. Define the parameters $h_\pm$ and ν by

$$2h_\pm = \frac{d}{2} \pm \nu; \qquad \nu = \frac{1}{2}\sqrt{d^2 + 4m^2}\, R^2. \qquad (4)$$

Normalizable solutions with definite angular momenta are of the form

$$\phi_{nl\vec{m}} = e^{-i\omega_{nl}t} Y_{l\vec{m}} \chi_{nl}(\rho) \qquad (5)$$

and have asymptotic behavior

$$\chi_{nl}(\rho) \overset{\rho\to\pi/2}{\longrightarrow} k_{nl}(\cos\rho)^{2h_+} \qquad (6)$$

for some constants k_{nl}. Explicit formulas for these solutions are given in [10]. The discrete eigenfrequencies are

$$\omega_{nl} = 2h_+ + 2n + l, \qquad n = 0, 1, 2, \ldots. \qquad (7)$$

We will write non-normalizable solutions as

$$\phi_{\omega l\vec{m}} = e^{-i\omega t} Y_{l\vec{m}} \chi_{\omega l}(\rho). \qquad (8)$$

These have asymptotic behavior

$$\chi_{\omega l} \overset{\rho\to\pi/2}{\longrightarrow} (\cos\rho)^{2h_-}, \qquad (9)$$

where a convenient normalization convention has been chosen by fixing the overall constant.

We will also require some assumptions about the spectrum. This is classified according to the representations of $SO(d,2)$. A given representation is determined by its weight Δ (which corresponds to the conformal weight in the CFT) and contains states $|\Delta; n, l, \vec{m}\rangle$. Here n is the principal quantum number and $l, \vec{m}$ are the standard angular quantum numbers, as above. The energy, defined with respect to global time, is given by

$$\omega_{n\ell}^{\Delta} = \Delta + 2n + l\,. \tag{10}$$

For a free field,

$$\Delta = 2h_+\,. \tag{11}$$

We assume that the states of the interacting scalar theory consist of the vacuum $|0\rangle$, the single particle states $|\Delta; n, l, \vec{m}\rangle$, and multiparticle states $|\Delta_\beta; n, l, \vec{m}\rangle_\beta$, where β is an additional state label. For the interacting field, Δ may be renormalized and is not necessarily given in terms of the bare mass by (11). We also assume that

$$\Delta_\beta > \Delta \tag{12}$$

for all multiparticle states.

Some useful properties of the bulk-boundary propagator are also needed. Suppose that we seek a solution of the free equation (3) satisfying the boundary condition

$$\phi \overset{\rho\to\pi/2}{\longrightarrow} (\cos\rho)^{2h_-} f(b)\,, \tag{13}$$

where $b = (t, \Omega)$ denotes the boundary coordinates and f is some specified boundary value. Witten [3] defines the bulk-boundary Green function to be the kernel that provides the solution:

$$\phi(x) = \int db\, f(b) G_{B\partial}(b,x)\,. \tag{14}$$

(Although Witten's definition was made in the euclidean continuation of AdS, the formalism naturally extends to Lorentzian signature, as discussed in Refs. [4,11].) Explicit expressions for $G_{B\partial}$ can then be found [3] using the resulting condition that $G_{B\partial}$ must asymptote to a delta function at the boundary.

It is easy to derive an alternate formula for $G_{B\partial}$ in terms of the bulk Feynman propagator $G_B(x,x')$ using an AdS variant of the usual Green's theorem argument. Consider the solution ϕ with the above boundary conditions (13). Define a region V by $\rho < \bar{\rho} \approx \pi/2$. Using

$$(\Box_x - m^2) G_B(x,x') = -\delta(x,x')\,, \tag{15}$$

we may rewrite ϕ as

$$\phi(x') = -\int_V dV\, \phi(x) (\Box_x - m^2) G_B(x,x') \tag{16}$$

and then integrate twice by parts to find

$$\phi(x') = -\int_{\partial V} d\Sigma^\mu\, \phi(x) \overleftrightarrow{\partial}_\mu G_B(x,x')\,. \tag{17}$$

At the boundary the Feynman propagator scales as

$$G_B(x,x') \overset{\rho\to\pi/2}{\longrightarrow} (\cos\rho)^{2h_+} G(b,x') \tag{18}$$

for some function G. Substituting this and the boundary behavior (13) into (17) gives

$$\phi(x') = -R^{d-1} \int db\, f(b) G(b,x') \lim_{\rho\to\pi/2} (\tan\rho)^{d-1} [(\cos\rho)^{2h_-} \overleftrightarrow{\partial}_\rho (\cos\rho)^{2h_+}]\,, \tag{19}$$

with limit

$$\phi(x') = 2\nu R^{d-1} \int db\, f(b) G(b,x')\,. \tag{20}$$

Comparison with (14) then shows

$$G_{B\partial}(b,x') = 2\nu R^{d-1} \lim_{\rho\to\pi/2} (\cos\rho)^{-2h_+} G_B(x,x')\,, \tag{21}$$

in agreement with [12].

Notice that (3) and (13) do not uniquely specify the solution ϕ; one may always add a normalizable mode without modifying the boundary behavior (13). Specifically, suppose that $f(b)$ falls to zero in the far past and future. Then, in the asymptotic past and future, ϕ must be a linear combination of the normalizable modes (5). With the preceding construction of $G_{B\partial}$, Eq. (14) gives the solution that is purely positive frequency in the far future and purely negative frequency in the far past. Other solutions can be obtained by modifying the temporal boundary conditions on the bulk Green function, e.g., by using retarded or advanced propagators.

Note that the non-normalizable solutions $\phi_{\omega l \vec{m}}$ can be recovered from (14); in the limit $f \to e^{-i\omega t} Y_{l\vec{m}}$, (9) coincides with (13). Therefore

$$\phi_{\omega l\vec{m}}(x) = \int db\, e^{-i\omega t} Y_{l\vec{m}} G_{B\partial}(b,x) \equiv G_{B\partial}(-\omega, l, -\vec{m}; x)\,. \tag{22}$$

For a general f, we can therefore rewrite (14) in terms of the Fourier transform $f_{l\vec{m}}(\omega)$ as

$$\phi_f(x) = \int db\, f(b) G_{B\partial}(b,x) = \sum_{l,\vec{m}} \int \frac{d\omega}{2\pi} f_{l\vec{m}}(\omega) \phi_{\omega l\vec{m}}(x)\,. \tag{23}$$

For appropriately chosen f, this function defines a solution corresponding to a wave packet. The function f determines the packet profile.

These can be thought of as packets incident from infinity in AdS. Notice that, according to (9), they typically diverge at the boundary. There is a physical reason for this: Motion in the region near the boundary is classically forbidden. Therefore the amplitude for a particle incident from infinity to reach the center of AdS is suppressed by an infinite tunneling factor. However, the amplitude for a particle to reach the center of AdS may be kept finite by rescaling the wave function such that the incident amplitude at infinity is infinite.

One concrete way to think of this is to imagine cutting off AdS at a large but finite radius and patching the resulting AdS bubble into a spacetime with a bona fide null infinity, as in [4]. A beam of particles from this asymptotic space can be focused to collide with another beam in the center of the AdS region. Most of the incident flux is reflected off the potential barrier resulting from the AdS geometry, so in order for the beams to penetrate to the center the incident amplitudes must be large. The wave-packet definitions above, which are given intrinsically in AdS without reference to an auxiliary bubble picture, can be thought of as arising from the limit where the radius of the AdS bubble goes to infinity while simultaneously scaling up the incident beam amplitudes.

These wave packets can now be used to construct operators that create in and out states. (References [5,6] outline the construction of such operators in the infinite-N limit. Here we will explicitly construct such operators for arbitrary N.) These asymptotic operators will be defined by

$$\alpha_f = \lim_{\bar\rho\to\pi/2} \int_\Sigma d\Sigma^\mu\, \phi_f^* \overleftrightarrow{\partial}_\mu \Phi\,, \tag{24}$$

where $\Sigma = \partial V$ for the region V defined above and Φ is the full *interacting* field. We also define the plane wave limit of these operators,

$$\alpha_{\omega l\vec m} = \lim_{\bar\rho\to\pi/2} \int_\Sigma d\Sigma^\mu\, \phi_{\omega l\vec m}^* \overleftrightarrow{\partial}_\mu \Phi\,. \tag{25}$$

If Φ is replaced by the free field,

$$\phi = \sum_{n,l,\vec m} a_{nl\vec m} e^{-i\omega_{nl}t}\phi_{nl\vec m} + a_{nl\vec m}^\dagger e^{i\omega_{nl}t}\phi_{nl\vec m}^*\,, \tag{26}$$

then (25) gives

$$\alpha_{\omega l\vec m} = -4\pi\nu R^{d-1}\sum_n k_{nl}[\delta(\omega - \omega_{nl})a_{nl\vec m} + \delta(\omega + \omega_{nl})a_{nl,-\vec m}^\dagger]\,, \tag{27}$$

where the k_{nl} appeared in (6). This suggests that the positive and negative frequency $\alpha_{\omega l\vec m}$'s can be thought of as annihilation and creation operators, respectively.

This is confirmed by the following critical relations, which hold for the operators constructed from the full interacting field:

$$\langle 0|\alpha_f|\Delta; n, l, \vec m\rangle = -2\nu R^{d-1}N(\Delta)k_{nl}f_{l\vec m}^*(\omega_{nl})\,, \tag{28}$$

$$\langle \Delta; n, l, \vec m|\alpha_f|0\rangle = -2\nu R^{d-1}N(\Delta)k_{nl}f_{l,-\vec m}^*(-\omega_{nl})\,, \tag{29}$$

and

$$\langle 0|\alpha_f|\Delta_\beta; n', l', \vec m'\rangle_\beta = 0\,, \tag{30}$$

where $N(\Delta)$ is another constant. Therefore α_f with positive-frequency f annihilates a particle at the boundary, and with negative-frequency f creates a particle at the boundary. Furthermore, Eq. (30) implies that the α_f's only annihilate/create *single* particle states.

The first of these relations is proved by recalling that, by symmetry, the full interacting field must satisfy [13]

$$\langle 0|\Phi(x)|; n, l, \vec m\rangle = N(\Delta)\phi_{nl\vec m}(x) \tag{31}$$

for some normalization factor $N(\Delta)$. Then the definition (25) and a derivation such as that in (19) and (20) immediately gives (28). Note that in order for this to be true the modes in (23) and (25) must be defined with the mass fixed by $2h_+ = \Delta$, corresponding to using the renormalized physical mass of the single particle state. Analogous reasoning proves Eq. (29).

Equation (30) is shown by noting that, again purely from the SO$(d,2)$ symmetry,

$$\langle 0|\Phi(x)|\Delta_\beta; n, l, \vec m\rangle_\beta = N_\beta(\Delta_\beta)\phi_{nl\vec m}^{\Delta_\beta}(x)\,, \tag{32}$$

where $\phi_{nl\vec m}^{\Delta_\beta}$ is defined using the mass parameter corresponding to the multiparticle Δ_β. Again, the matrix element (30) can be found from reasoning parallel to (19) and (20), but now the result contains

$$\lim_{\rho\to\pi/2}(\cos\rho)^{\Delta_\beta - \Delta}. \tag{33}$$

This vanishes by Eq. (12).

In and out states are now readily defined. For positive-frequency functions f_i, define

$$\alpha_{f_i}^{\rm in} = \alpha_{f_i}/Z_i\,, \tag{34}$$

and for negative-frequency functions f_j', define

$$\alpha_{f_j'}^{\rm out\dagger} = \alpha_{f_j'}/Z_j'\,, \tag{35}$$

where the Z_i are wave-function renormalization factors necessary to cancel normalization constants such as those in (28) and (29). The in and out states are then

$$|f_i\rangle_{\rm in} = \prod_i \alpha_{f_i}^{\rm in\dagger}|0\rangle \tag{36}$$

and

$$_{\rm out}\langle f_j'| = \langle 0|\prod_j \alpha_{f_j'}^{\rm out}\,. \tag{37}$$

These states in turn lead to construction of the boundary S matrix. Suppose that the wave packets f_i, f_j' are nonoverlapping, and that the support of all of the f_j''s lies to the future of that of all of the f_i's. The boundary S matrix is then defined as

$$S_\partial[f_1\cdots f_m; f_1'\cdots f_n'] \equiv \left\langle 0|T\prod_j \alpha_{f_j'}^{\rm out}\prod_i \alpha_{f_i}^{\rm in\dagger}|0\right\rangle. \tag{38}$$

Although the interpretation is less transparent, the same definition can be adopted for f_i and f_j' not satisfying the above conditions.

In flat space, the S matrix is related by the LSZ formula to truncated correlation functions. A similar formula can now be derived for the boundary S matrix, which will be given in terms of bulk correlation functions. Consider a finite region V' defined such as V above, but with boundaries Σ_{-T}, Σ_T of constant time $\pm T$ lying to the far past and far future of the wave packets' support. The Gauss theorem applied to (24) gives

$$\alpha_f^{\mathrm{in}\dagger} = \frac{1}{Z} \lim_{\bar\rho\to\pi/2} \left[\int_{V'} dV\, \nabla^\mu (\phi_f \overleftrightarrow{\partial}_\mu \Phi) - \int_{\Sigma_{-T}+\Sigma_T} d\Sigma^\mu\, \phi_f \overleftrightarrow{\partial}_\mu \Phi \right], \tag{39}$$

where the $T \to \infty$ limit is also understood. The surface term at Σ_{-T} vanishes because ϕ_f is positive frequency and therefore vanishes in the far past. Inside (38), the time ordering takes the surface term at Σ_T to the left. We can then insert a complete set of states $|s\rangle$ to find an expression of the form

$$\sum_s \int_{\Sigma_T} d\Sigma^\mu\, \phi_f \overleftrightarrow{\partial}_\mu \langle 0|\Phi|s\rangle \langle s|\psi\rangle\,. \tag{40}$$

This vanishes by Eq. (31). The bulk term is left and, after using the free equation for ϕ_f and taking $T \to \infty$, becomes

$$\alpha_f^{\mathrm{in}\dagger} \approx \frac{1}{Z} \int dV\, \phi_f(\Box - m^2)\Phi\,, \tag{41}$$

where ($\approx$) denotes equality inside (38). Similar arguments hold for α_f^{out}. The LSZ formula immediately follows:

$$S_\partial[f_1\cdots f_m; f_1'\cdots f_n'] = \int \prod_i \left[\frac{dV_i}{Z_i}\, \phi_{f_i}(x_i) \right] \prod_j \left[\frac{dV_j'}{Z_j'}\, \phi_{f_j'}(x_j') \right] \langle 0|T \prod_i \Phi(x_i) \prod_j \Phi(x_j')\,|0\rangle_T\,. \tag{42}$$

Here the kinetic operator in (41) has removed the external legs, and the "T" subscript denotes the resulting truncated Green function.

The relation to correlation functions in the conformal field theory now follows trivially. Witten [3] has shown that the CFT correlators are given in terms of the truncated bulk correlators and the bulk-boundary propagator as

$$\left\langle T \prod_a \mathcal{O}(b_a) \right\rangle = \int \prod_a [dV_a G_{B\partial}(b_a, x_a)] \langle 0|T \prod_a \Phi(x_a)\,|0\rangle_T\,. \tag{43}$$

After substituting (23) into (42), S_∂ is therefore given by

$$S_\partial[f_1\cdots f_m; f_1'\cdots f_n'] = \int \prod_i \left[db_i\, \frac{f_i(b_i)}{Z_i} \right] \int \prod_j \left[db_j'\, \frac{f_j'(b_j')}{Z_j'} \right] \left\langle T \prod_i \mathcal{O}(b_i) \prod_j \mathcal{O}(b_j') \right\rangle, \tag{44}$$

or in the plane wave limit $f \to e^{-i\omega t} Y_{l\vec m}$,

$$S_\partial[\{\omega_i, l_i, \vec m_i\}; \{\omega_j', l_j', \vec m_j'\}] = \left\langle T \prod_i \frac{1}{Z_i}\, \mathcal{O}(\omega_i, l_i, \vec m_i) \prod_j \frac{1}{Z_j'}\, \mathcal{O}(\omega_j', l_j', \vec m_j') \right\rangle. \tag{45}$$

These strikingly simple relations tell us that the CFT correlators directly determine the boundary S matrix, and thus provide an extremely simply dictionary between scattering amplitudes in anti–de Sitter space and the conformal field theory correlation functions.

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