Asymptotic Theory for the Probability Density Functions in Burgers Turbulence

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(Received 10 December 1998; revised manuscript received 24 June 1999)

A systematic analysis is carried out for the randomly forced Burgers equation in the infinite Reynolds number (inviscid) limit. No closure approximations are made. Instead the probability density functions of velocity and velocity gradient are related to the statistics of quantities defined along the shocks. This method allows one to compute the dissipative anomalies, as well as asymptotics for the structure functions and the probability density functions. It is shown that the left tail for the probability density function of the velocity gradient has to decay faster than $|\xi|^{-3}$. A further argument confirms the prediction of E *et al.* [Phys. Rev. Lett. **78**, 1904 (1997)] that it should decay as $|\xi|^{-7/2}$.

PACS numbers: 47.27.Gs, 02.50.Ey, 05.40.–a

In this Letter, we focus on statistical properties of solutions of the randomly forced Burgers equation,

$$u_t + uu_x = \nu u_{xx} + f\,, \tag{1}$$

where f is a zero-mean, statistically homogeneous, white-in-time Gaussian process with covariance,

$$\langle f(x,t) f(y,s)\rangle = 2B(x-y)\delta(t-s)\,, \tag{2}$$

where $B(x)$ is smooth. Equation (1) and its multidimensional versions have received much attention recently for two main reasons. First, (1) serves as a qualitative model for a wide variety of problems including charge density waves [1], vortex lines in high temperature superconductors [2], dislocations in disordered solids, kinetic roughening of interfaces in epitaxial growth [3], etc. The second reason is that (1) has served as the benchmark for approximation developed for solving the problem of hydrodynamic turbulence. This role of (1) is made more evident by the recent flourish of activities introducing fairly sophisticated techniques in field theory to hydrodynamics [4–6]. Since the phenomenology of the so-called Burgers turbulence is far simpler than that of real turbulence, one hopes that exact results can be obtained which can then be used to benchmark the methods. However, thus far our experience has proved otherwise: The problem of forced Burgers turbulence is complicated enough that a wide variety of predictions have been made as a consequence of the wide variety of techniques used [5–13]. The main purpose of the present Letter is to clarify this situation and obtain exact results that are expected for forced Burgers turbulence.

We are particularly interested in the probability density function (PDF) of the velocity gradient $\xi(x,t) = u_x(x,t)$, since it depends heavily on the intermittent events created by the shocks. Assuming statistical homogeneity, and letting $Q(\xi,t)$ be the PDF of $\xi(x,t)$, it can be shown that Q satisfies

$$Q_t = \xi Q + (\xi^2 Q)_\xi + B_1 Q_{\xi\xi} - \nu(\langle \xi_{xx} \,|\, \xi\rangle Q)_\xi\,, \tag{3}$$

where $B_1 = -B_{xx}(0)$. $\langle \xi_{xx} \,|\, \xi\rangle$ is the ensemble average of ξ_{xx} conditional on ξ. The explicit form of this term is unknown, leaving (3) unclosed. There have been several proposals on how to evaluate $-\nu(\langle \xi_{xx} \,|\, \xi\rangle Q)_\xi$ approximately in the infinite Reynolds number (inviscid) limit:

$$F(\xi,t) = -\lim_{\nu\to 0} \nu(\langle \xi_{xx} \,|\, \xi\rangle Q)_\xi\,. \tag{4}$$

At steady state, they all lead to an asymptotic expression of the form

$$Q(\xi) \sim \begin{cases} C_-|\xi|^{-\alpha} & \text{as } \xi \to -\infty \\ C_+\xi^{\beta} e^{-\xi^3/(3B_1)} & \text{as } \xi \to +\infty\,, \end{cases} \tag{5}$$

for Q, but with a variety of values for the exponents α and β (here the $C_\pm$'s are constants). By invoking the operator product expansion, Polyakov [5] suggested that $F = aQ + b\xi Q$, with $a = 0$ and $b = -1/2$. This leads to $\alpha = 5/2$ and $\beta = 1/2$. Boldyrev [9] considered the same closure with $-1 \le b \le 0$, which gives $2 \le \alpha \le 3$ and $\beta = 1 + b$. Bouchaud and Mézard [7] introduced a Langevin equation for the local slope of the velocity, which gives $2 \le \alpha \le 3, \beta = 0$. The instanton analysis [6,8] predicts the right tail of Q without giving a precise value for β, and it does not give any specific prediction for the left tail. E *et al.* [10] made a geometrical evaluation of the effect of F, based on the observation that large negative gradients are generated near shock creation. Their analysis gives a rigorous upper bound for α: $\alpha \le 7/2$. In [10], it was claimed that this bound is actually reached, i.e., $\alpha = 7/2$. Finally Gotoh and Kraichnan [11] argued that the viscous term is negligible to leading order for large $|\xi|$, i.e., $F \approx 0$ for $|\xi| \gg B_1^{1/3}$. This approximation leads to $\alpha = 3$ and $\beta = 1$. In this Letter, we proceed at an exact evaluation of (4) and we prove that α has to be strictly larger than 3 (a result which holds not only at steady state). At steady state, we prove that $\beta = 1$ and we give an argument which supports strongly the prediction of [10], namely, $\alpha = 7/2$.

To begin with, let us remark that it is established in the mathematics literature that the infinite Reynolds number

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limit,

$$u^0(x,t) = \lim_{\nu\to 0} u(x,t), \tag{6}$$

exists for almost all (x,t) (see, e.g., [14]). Since u^0 will, in general, be discontinuous due to the presence shocks, the inviscid Burgers equations have to be interpreted in the weak sense by requiring

$$\iint dx\, dt\{u^0\varphi_t + \tfrac{1}{2}(u^0)^2\varphi_x + f\varphi\} = 0, \tag{7}$$

for all test functions φ. The solutions u^0 satisfying (7) are called weak solutions. In this framework, the effect of dissipation is accounted for by jump (or entropy) conditions at the shocks. An alternative, more intuitive, way of accessing the effect of the viscous shock on the velocity profile outside the shock is to carry out an asymptotic analysis near and inside the shock. Here, we will take the second approach and refer the interested reader to [15] for calculations with weak solutions. It is important to remark that the two approaches lead to the same results.

Before considering velocity gradient, it is helpful to study the statistics of velocity itself. Let $R(u,t)$ be the PDF of $u(x,t)$. Assuming statistical homogeneity, R satisfies

$$R_t = B_0 R_{uu} - \nu(\langle u_{xx} \,|\, u\rangle R)_u, \tag{8}$$

where $B_0 = B(0)$. To compute $-\nu(\langle u_{xx} \,|\, u\rangle R)_u$, let us note that for $\nu \ll 1$, the solutions of (1) consist of smooth pieces where the viscous effect is negligible, separated by thin shock intervals inside which the viscous effect is important. In these intervals, boundary layer analysis can be used to approximate $u(x,t)$. The basic idea is to split u into the sum of an inner solution near the shock and an outer solution away from the shock, and use systematic matched asymptotics to construct a uniform approximation of u.

The outer solution can be obtained as a series expansion in ν: $u = u^{\text{out}} = u_0 + \nu u_1 + O(\nu^2)$. In order to deal with the inner solution around the shock, say, at $x = y$, define the stretched variable $z = (x - y)/\nu$, and let $u(x,t) = u^{\text{in}}(x,t) = v[(x-y)/\nu, t]$. The definition of the stretched variable is motivated by the fact that the width of the shock is $O(\nu)$. We look for an expression for v in the form of a series expansion in ν: $v = v_0 + \nu v_1 + O(\nu^2)$. Following a standard matched asymptotics procedure, we can systematically write down equations for $v_0, v_1, \ldots.$ For example, $v_0(z,t)$ satisfies $(v_0 - \bar{u})v_{0z} = v_{0zz}$, yielding the leading order velocity profile inside the shock layer: $v_0(z,t) = \bar{u} - (s/2)\tanh(sz/4)$. Here, $\bar{u} = dy/dt$, s is the jump across the shock, and the actual values of $\bar{u}$ and s are obtained from the matching conditions between $u^{\text{out}} \approx u_0$ and $u^{\text{in}} \approx v_0$. It is well known that $s \leq 0$. Higher order corrections can be obtained in a similar way (for details, see [13,15]).

We use the results of the boundary layer analysis to evaluate the viscous term in (8). By definition [16],

$$\nu\langle u_{xx} \,|\, u\rangle R = \nu \lim_{L\to\infty} \frac{1}{2L}\int_{-L}^{L} dx\, u_{xx}\delta[u - u(x,t)]. \tag{9}$$

In the limit as $\nu \to 0$ only small intervals around the shocks will contribute to the integral, and in these intervals we can approximate the velocity by u^{in}. To $O(\nu)$, this gives

$$\nu\langle u_{xx} \,|\, u\rangle R = \nu \lim_{L\to\infty} \frac{N}{2L}\frac{1}{N}\sum_j \int_{j\text{th layer}} dx\, u_{xx}^{\text{in}}\delta[u - u^{\text{in}}(x,t)] = \rho\int ds\, d\bar{u}\, T(\bar{u},s,t)\int_{-\infty}^{+\infty} dz\, v_{0zz}\delta[u - v_0(z,t)], \tag{10}$$

where in the second integral we changed to the stretched variable $z = (x-y)/\nu$ and took $L \to \infty$. Here, N denotes the number of shocks in $[-L, L]$, $\rho = \lim_{L\to\infty} N/2L$ is the number density of shocks, $T(\bar{u},s,t)$ is the PDF of $[\bar{u}(y,t), s(y,t)]$ conditional on the property that there is a shock at position y (T is independent of y because of statistical homogeneity). To evaluate the z integral in (10), we can use the equation for v_0, $(v_0 - \bar{u})v_{0z} = v_{0zz}$, and change the integration variable from z to v_0 using $dz v_{0zz} = dv_0 v_{0zz}/v_{0z} = dv_0(v_0 - \bar{u})$. The result is

$$\lim_{\nu\to 0}\nu\langle u_{xx} \,|\, u\rangle R = -\rho\int_{-\infty}^{0} ds\int_{u+s/2}^{u-s/2} d\bar{u}(u - \bar{u})T(\bar{u},s,t). \tag{11}$$

This equation gives an *exact* expression for the viscous contribution in the limit as $\nu \to 0$ in terms of certain statistical quantities associated with the shocks. Of course, using (11) in (8) does not lead to a closed equation since T remains to be specified. However, information can already be obtained at this point without resorting to any closure assumption. For instance, using (11) in (8) and taking the second moment of the resulting equation yields $d\langle u^2\rangle/dt = 2B_0 - 2\epsilon$ with

$$\epsilon = \lim_{\nu\to 0}\nu\langle u_x^2\rangle = \frac{1}{12}\rho\langle |s|^3\rangle. \tag{12}$$

This gives a quantitative description of the energy dissipation at the shocks. At statistical steady state, this gives $B_0 = \rho\langle |s|^3\rangle/12$.

Similar calculations can be carried out for multipoint PDF's and, in particular, for $Z^\delta(w,x,t)$, the PDF of the velocity difference $\delta u(x,z,t) = u(x+z,t) - u(z,t)$, $x > 0$. It leads to an equation of the form

$$Z_t^\delta = -wZ_x^\delta - 2\int_{-\infty}^{w} dw'\, Z_x^\delta(w',x,t) + 2[B_0 - B(x)]Z_{ww}^\delta + G^\delta(w,x,t), \tag{13}$$

where, to $o(1)$, G^δ is given by

$$G^\delta(w,x,t) = \rho[wS(w,t) + \langle s\rangle\delta(w)] - 2\rho H(w) + 2\rho\int_{-\infty}^{w} dw'\, S(w',t) + o(1). \qquad (14)$$

Here, $H(w)$ is the Heaviside function and $S(s,t) = \int d\bar{u}\, T(\bar{u},s,t)$ is the conditional PDF of $s(y,t)$. By direct substitution, it may be shown that the solution of (13) is [17]

$$Z^\delta = (1-\rho x)\frac{1}{x}Q\left(\frac{w}{x},t\right) + \rho x S(w,t) + o(x). \qquad (15)$$

The first term in this expression contains $Q(\xi,t)$, the PDF of the nonsingular part of the velocity gradient, to be considered below [see (18)]. This term accounts for the realizations of the flow where there is no shock between z and $x+z$ [an event of probability $1-\rho x + O(x^2)$]. The next term in (15), $\rho x S(w,t)$, accounts for the realizations of the flow where there is a shock between z and $x+z$ [an event of probability $\rho x + O(x^2)$]. Equation (15) can be used to compute the structure functions, $\langle|\delta u|^a\rangle = \int dw |w|^a Z^\delta$. This gives

$$\langle|\delta u|^a\rangle = \begin{cases} x^a\langle|\xi|^a\rangle + o(x^a) & \text{if } 0 \le a < 1 \\ x\rho\langle|s|^a\rangle + o(x) & \text{if } 1 < a, \end{cases} \qquad (16)$$

where $\langle|\xi|^a\rangle = \int d\xi |\xi|^a Q$. Using $\rho\langle|s|^3\rangle = 12B_0$, we get Kolmogorov's relation for $a = 3$:

$$\langle|\delta u|^3\rangle = 12xB_0 + o(x). \qquad (17)$$

We now go back to the velocity gradient. Observe first that, in the limit as $\nu \to 0$, the velocity gradient can be written as

$$u_x(x,t) = \xi(x,t) + \sum_j s(y_j)\delta(x-y_j), \qquad (18)$$

where the y_j's are the locations of the shocks, and ξ is the *regular* part of u_x. Assuming homogeneity, a direct consequence of (18) is

$$\langle u_x\rangle = \langle\xi\rangle + \rho\langle s\rangle = 0. \qquad (19)$$

Unlike the viscous case where $\xi = u_x$, hence, $\langle\xi\rangle = 0$, we have $\langle\xi\rangle = -\rho\langle s\rangle \neq 0$ in the limit as $\nu \to 0$. Note also that the solutions of (3) converge as $\nu \to 0$ to the PDF of ξ only, which is still going to be denoted by Q.

To evaluate F, there are two ways to proceed. One is to rewrite (13) in terms of the PDF of $[u(x+z,t) - u(z,t)]/x$ and take the limit as x goes to zero. This is the approach taken in [15]. The other is to evaluate (4) directly. The two approaches amount to different orders of taking the limit $x \to 0, \nu \to 0$, and give the same result. Hence, the two limiting processes commute. We will take the second approach and evaluate (4) using the same basic idea as above. By definition [16],

$$\nu\langle\xi_{xx}\,|\,\xi\rangle Q = \nu\lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L} dx\,\xi_{xx}\delta[\xi - \xi(x,t)]. \qquad (20)$$

This integral is evaluated similarly as (9). We use boundary layer analysis to approximate $\xi(x,t)$ by the inner solution $\xi^{\rm in}(x,t) = u_x^{\rm in}(x,t)$ in the intervals around the shocks which give the only surviving contribution in the limit as $\nu \to 0$. We skip these calculations (for details, see [13,15]), noting simply that the boundary layer analysis has to be carried one order further for the velocity gradient than for velocity itself. Indeed $\xi^{\rm in} = u_x^{\rm in} = \nu^{-1}v_{0z} + v_1 + O(\nu)$, and the contribution of v_{1z} turns out to be important. The calculation eventually leads to the following expression for $F = -\lim_{\nu\to 0}\nu(\langle\xi_{xx}\,|\,\xi\rangle Q)_\xi$:

$$F(\xi,t) = \rho\int_{-\infty}^{0} ds\, sV(s,\xi,t), \qquad (21)$$

where $V(s,\xi,t) = \frac{1}{2}[V_+(s,\xi,t) + V_-(s,\xi,t)]$, $V_\pm(s,\xi_\pm,t)$ are the conditional PDF's of $[s(y,t),\xi_\pm(y,t)]$, $\xi_\pm(y,t)$ are the velocity gradients at the left and at the right of the shock. Thus, in the infinite Reynolds number limit, the PDF of the regular part of the velocity gradient satisfies

$$Q_t = \xi Q + (\xi^2 Q)_\xi + B_1 Q_{\xi\xi} + F(\xi,t). \qquad (22)$$

One important consequence of (22), together with the equations of motion along the shocks,

$$\begin{aligned} \frac{ds}{dt} &= -\frac{s}{2}(\xi_+ + \xi_-), \\ \frac{d\bar{u}}{dt} &= -\frac{s}{4}(\xi_+ - \xi_-) + f, \end{aligned} \qquad (23)$$

is the statement that

$$\lim_{|\xi|\to+\infty}\xi^3 Q(\xi,t) = 0, \qquad (24)$$

i.e., Q goes to zero *faster* than $|\xi|^{-3}$ as $\xi \to -\infty$ and $\xi \to +\infty$, and $\alpha > 3$ in (5). To see this, take the first moment of (22):

$$\frac{d}{dt}\langle\xi\rangle = [\xi^3 Q]_{-\infty}^{+\infty} + \frac{\rho}{2}(\langle s\xi_-\rangle + \langle s\xi_+\rangle), \qquad (25)$$

where we used $\int d\xi\,\xi F = \rho(\langle s\xi_-\rangle + \langle s\xi_+\rangle)/2$. Next, average the first equations in (23):

$$\frac{d}{dt}(\rho\langle s\rangle) = -\frac{\rho}{2}(\langle s\xi_-\rangle + \langle s\xi_+\rangle). \qquad (26)$$

This equation uses the fact that shocks are created at zero amplitude, and shock strengths add up at collision. These are consequences of the fact that the forcing is smooth in space. Since $d\langle\xi\rangle/dt = -d(\rho\langle s\rangle)/dt$ from (19), the comparison between (25) and (26) tells us that the boundary term in (25) must be zero. Since $Q \ge 0$, $\xi^3 Q$ has different sign for large positive and large negative values of ξ. Therefore we must have $\lim_{\xi\to+\infty}\xi^3 Q = 0$ *and* $\lim_{\xi\to-\infty}\xi^3 Q = 0$. Hence, (24).

The analysis can be carried out one step further for the stationary case ($Q_t = 0$). Then, treating (22) as an inhomogeneous second order ordinary differential equation, we can write its general solution as $Q = C_1 Q_1 + C_2 Q_2 + Q_3$, where C_1 and C_2 are constants, Q_1 and Q_2 are two linearly independent solutions of the homogeneous equation associated with (22), and Q_3 is some particular solution of this equation. One such particular solution is

$$Q_3 = \int_{-\infty}^{\xi} d\xi' \,\frac{\xi' F(\xi')}{B_1} - \frac{\xi \mathrm{e}^{-\Lambda}}{B_1} \int_{-\infty}^{\xi} d\xi' \,\mathrm{e}^{\Lambda'} G(\xi'), \tag{27}$$

where $\Lambda = \xi^3/(3B_1)$ and

$$G(\xi) = F(\xi) + \xi \int_{-\infty}^{\xi} d\xi' \,\frac{\xi' F(\xi')}{B_1}. \tag{28}$$

With this particular solution, it can be shown (see [13,15] for details) that the realizability constraints imply that $C_1 = C_2 = 0$, i.e., the only non-negative, integrable solution is $Q = Q_3$. Furthermore, in order that Q actually be non-negative, F must satisfy

$$\lim_{\xi \to +\infty} \xi^{-2} \mathrm{e}^{\Lambda} F(\xi) = 0. \tag{29}$$

Substituting into (27), we get

$$Q \sim \begin{cases} C_- |\xi|^{-3} \int_{-\infty}^{\xi} d\xi' \,\xi' F(\xi') & \text{as } \xi \to -\infty \\ C_+ \xi \mathrm{e}^{-\Lambda} & \text{as } \xi \to +\infty, \end{cases} \tag{30}$$

which confirms the result $Q \sim C_- |\xi|^{-\alpha}$ with $\alpha > 3$ as $\xi \to -\infty$, and gives $\beta = 1$.

The actual value of the exponent α depends on the asymptotic behavior of F. The latter can be obtained from further considerations on the dynamics of the shock (23). This is rather involved and will be left to [15]. The result gives $\alpha = 7/2$ which confirms the prediction of [10]. Here, we will restrict ourselves to an interpretation of the current approach in terms of the geometric picture on the local analysis of shock creation [18]. Observe that the largest values of $\xi_\pm$ are achieved just after the shock formation. For a shock created at $(x, t) = (0, 0)$ with velocity $u = 0$, we have locally $x = ut - au^3 + \cdots$. It follows that $s = -2(t/a)^{1/2}$, $\xi_\pm = -1/2t$. Assuming that these give the dominant contribution to $F(\xi)$ for large negative values of ξ, the asymptotic form of F is $F \sim C \int_0^\infty dt\, s[\delta(\xi - \xi_-) + \delta(\xi - \xi_+)]$, where C is some constant related to the statistics of the shock lifetime and a, $s = -2(t/a)^{1/2}$, $\xi_\pm = -1/2t$. Direct evaluation of this integral gives $F \sim C|\xi|^{-5/2}$, and, hence,

$$Q \sim C_- |\xi|^{-7/2} \quad \text{as } \xi \to -\infty. \tag{31}$$

Even though this argument gives only a lower bound for F at large negative values of ξ, further arguments presented in [15] indicate that this lower bound is actually sharp.

In summary, we derived master equations by evaluating the infinite Reynolds number limit of the dissipative effect of the singular structures, here the shocks. We also explored the consequences of the master equations without resorting to closure assumptions. Asymptotic scaling of the structure functions and bounds for the asymptotic behavior of the PDF of the velocity gradient were obtained using self-consistent asymptotic analysis and realizability constraints. Finally, the $|\xi|^{-7/2}$ prediction for the left tail of the velocity gradient PDF was obtained by local analysis around the singular structures. We certainly hope that this philosophy will be useful for other problems.

We thank S. A. Boldyrev, R. H. Kraichnan, A. P. Polyakov, and Ya. G. Sinai for stimulating discussions.

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