Stable Complexes of Parametrically Driven, Damped Nonlinear Schrödinger Solitons

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Since solitons of the parametrically driven damped nonlinear Schrödinger equation do not have oscillatory tails, it was suggested that they cannot form bound states. We show that this equation does support solitonic complexes, with the mechanism of their formation being different from the standard tail-overlap mechanism. One of the arising stationary complexes is found to be stable in a wide range of parameters, others unstable.

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Motivation.—Bound states of solitons and solitary pulses are attracting increasing attention in nonlinear optics [1–5], dynamics of fluids [6–9], and excitable media [10]. *Stable* bound states can compete with free solitons as alternative attractors. This is detrimental in nonlinear optics, for example, where the interaction between adjacent pulses poses limitations to the stable operation of transmission lines and information storage elements. *Unstable* solitonic complexes are not meaningless either; they serve as intermediate states visited by the system when in the spatiotemporal chaotic regime.

Here, we consider solitonic complexes in the parametrically driven damped nonlinear Schrödinger equation,

$$i\Psi_t + \Psi_{xx} + 2|\Psi|^2\Psi - \Psi = -i\gamma\Psi + h\overline{\Psi}. \qquad (1)$$

This equation describes the effect of phase-sensitive amplifiers in optics [11]: the nonlinear Faraday resonance in oscillating water troughs [8,9,12,13], convection in binary mixtures [14] and nematic liquid crystals [15], magnetization waves in easy-plane ferromagnets with a rf magnetic field in the easy plane [16], and synchronized oscillations in parametrically driven Frenkel-Kontorova chains [17].

Malomed noted that since solitons of Eq. (1) decay monotonically as $|x| \to \infty$, they cannot form bound states via the tail-overlap mechanism [3]. A variational analysis indicated that in the undamped case ($\gamma = 0$), a strong overlapping of solitons cannot lead to their binding either [7]. However, oscillatory and stationary soliton associations were observed in experiments with oscillating water troughs [8,9,18] and subsequently reproduced in numerical simulations of Eq. (1) [9].

These experiments and simulations have raised several challenging questions. Firstly and most importantly, an open problem is the very mechanism of the complex formation. Next, it was observed that stationary ("standing") complexes exist only at large separations [8]; it has therefore remained unclear whether these complexes are genuinely stationary or do diverge slowly due to an exponentially small repulsion. A related question is whether soliton associations can arise only on finite intervals under periodic boundary conditions and how essential the interval finiteness and periodicity are for their stability. (Note that the experiments of [8,9] were carried out on relatively short intervals [19]. On the other hand, periodic chains of solitons can form stable stationary states even in situations where a finite number of solitons do not bind [5].) Lastly, numerical simulations can detect only stable complexes; however, the description of the phase space is incomplete without knowledge of all unstable complexes and their bifurcations.

The purpose of this Letter is to answer some of these questions and gain insight into others. We focus on stationary complexes here. Oscillating complexes arise simply as Hopf bifurcations of the latter.

Variational approximation.—Two coexisting stationary solitons of Eq. (1) are given by

$$\Psi_\pm(x) = A_\pm e^{-i\theta_\pm}\,\mathrm{sech}(A_\pm x),$$
$$\cos 2\theta_\pm = \pm\sqrt{1 - \frac{\gamma^2}{h^2}}; \qquad A_\pm = \sqrt{1 + h\cos 2\theta_\pm}.$$

The soliton Ψ_- is always unstable [16] and hence is usually disregarded. We will attempt to approximate a complex of two solitons Ψ_+ by a trial function of the form

$$\Psi(x,t) = \psi(x - x_0)e^{ik(x-x_0)} + \psi(x + x_0)e^{-ik(x+x_0)}, \qquad (2)$$

where $\psi(x) = Ae^{-i\theta}\,\mathrm{sech}(Ax)$ and parameters x_0, k, θ, and A are allowed to depend on time. The evolution of the parameters can be found if one notices that Eq. (1) follows from the stationary action principle $\delta S = 0$, where $S = \int \mathcal{L}\, e^{2\gamma t}\, dt$ and the Lagrangian

$$\mathcal{L} = \mathrm{Re}\int (i\Psi_t\overline{\Psi} - |\Psi_x|^2 + |\Psi|^4 - |\Psi|^2 - h\Psi^2)\, dx. \qquad (3)$$

Feeding (2) into (3), integrating x out and denoting $z = 2Ax_0$, $p = 2k/A$ yields a four-dimensional Lagrangian

$$\mathcal{L} = 4A(1 - \sigma_s)\dot{\theta} + 2\eta\dot{A} + Ap\dot{z} - Az\sigma_s\dot{p} - H, \tag{4}$$

where the quantities σ and η are given by

$$\sigma_c(p,z) + i\sigma_s(p,z) = -\pi \frac{e^{ipz/2}}{\sinh(\pi p/2)\sinh z}; \qquad \eta(p,z) = \left[1 - \frac{\pi p/2}{\tanh(\pi p/2)} - \frac{z}{\tanh z}\right]\sigma_c + \frac{\pi\sigma_s \coth z}{\tanh(\pi p/2)}.$$

The Hamiltonian $H = H_0 + H_{\rm int}$ comprises the free soliton contribution,

$$H_0(\theta, A, p) = 4A - \frac{4}{3}A^3 + A^3p^2 + \frac{2\pi Ah\cos(2\theta)p}{\sinh(\pi p/2)},$$

and the Hamiltonian of the soliton-soliton interaction,

$$H_{\rm int}(\theta,A,p,z) = \frac{4\pi A^3}{\sinh(\pi p/2)}\left\{\left[\frac{2p\cos(pz/2)}{\sinh z}\right]_z - \left[\frac{\sin(pz/2)}{\sinh z}\right]_{zz} + \frac{4\sin(pz/2)}{\sinh^3 z} + \frac{\sin(pz/2)}{A^2\sinh z}\right\}$$
$$+ \frac{16A^3}{\sinh z}\left[\frac{z}{\sinh z}\right]_z + \frac{8\pi A^3}{\sinh(\pi p)\sinh z}\left[\frac{\sin(pz)}{\sinh z}\right]_z + \frac{4hAz}{\sinh z}\cos\left(\frac{pz}{2} + 2\theta\right).$$

The variations in z and θ yield, respectively,

$$2\gamma Ap + \partial_z H = 0, \tag{5}$$

$$8\gamma A(1 - \sigma_s) + \partial_\theta H = 0. \tag{6}$$

(Here and below we restrict ourselves to stationary solutions [20].) Since the second term in (5) decays rapidly as z grows, p has to be small. We expand all other variables in powers of p: $\theta = \sum\theta^{(n)}p^n$, $A = \sum A^{(n)}p^n$, $z = \sum z^{(n)}p^n$; $n = 0, 1, \ldots$. At the order p^0 equation (6) gives $\theta^{(0)} = \theta_+$, while the next order produces

$$h\sin(2\theta) = \gamma - p\frac{h\cos2\theta_+}{2}\frac{z^2}{z + \sinh z} + O(p^2). \tag{7}$$

Varying with respect to A we get $4\gamma\eta = -\partial_A H$. Noting that $\eta = O(p)$ and writing $H = \sum H^{(n)}(\theta, A, z)p^n$, the leading order is given by $\partial_A H^{(0)} = 0$, which amounts to

$$A^2 = \frac{(1 + h\cos2\theta_+)(z + \sinh z)}{\sinh z + 3z + 6[z(\cosh z - 3)/\sinh z]_z}. \tag{8}$$

This relation defines A as a monotonically growing function of z. Next, the variation with respect to p produces equation $2\gamma Az\sigma_s = \partial_p H$ whose leading order is

$$p\left\{A\frac{h\cos(2\theta_+)z^4}{\sinh z(z + \sinh z)} + 2H^{(2)}\right\} = 0, \tag{9}$$

where we have used Eq. (7). One readily checks that the expression in the curly brackets is linear in A^2 and hence Eq. (9) defines another function $A(z)$,

$$A^2\left[1 + \frac{(\pi^2 - 18)z + z^3}{6\sinh z} + \frac{(\pi^2 + 3z^2)\cosh z - 4(\pi^2 + 3z^2)}{3\sinh^2 z} + \frac{z(\pi^2 + z^2)(4\cosh z - 1)}{3\sinh^3 z}\right]$$
$$= h\cos2\theta_+\left[\frac{\pi^2}{6} + \frac{z^3}{2(z + \sinh z)}\right] + \frac{z(\pi^2 + z^2)}{6\sinh z}. \tag{10}$$

In Eqs. (8) and (10) A and z stand for $A^{(0)}$ and $z^{(0)}$. The curves (8) and (10) intersect at some point $(\tilde{z}, \tilde{A})$. For example, for $\gamma = 0.565$, $h = 0.9$ we have $\tilde{z} = 4.60$ and $\tilde{A} = 1.14$.

Finally, the stationary value of p is found from Eq. (5) where it is sufficient to keep terms up to p^1,

$$p(2\gamma A + \partial_z H_{\rm int}^{(1)}) + \partial_z H_{\rm int}^{(0)} = 0. \tag{11}$$

[Here we are regarding $\partial_z H_{\rm int}^{(0)}$ as a function of $z^{(0)}$, $A^{(0)}$, and $\theta^{(0)}$ and discarding p^1-corrections to this function which are negligible compared to the first term in (11).] From (11), the stationary value of p is

$$\tilde{p} = \frac{1}{2\gamma A}\frac{\partial_z H_{\rm int}^{(0)}}{1 - (z^2/\sinh z)_z}, \tag{12}$$

where $z = \tilde{z}$ and $A = \tilde{A}$. For example, for the above values $\gamma = 0.565$ and $h = 0.9$ Eq. (12) yields $\tilde{p} = -0.12$.

Thus, our approximate analysis predicts the existence of a stationary bound state of two solitons Ψ_+ on the infinite line. Below this complex (denoted $\Psi_{(++)}$) is reobtained numerically and Fig. 1 compares it to the variational approximation (2). It is seen that $\tilde{z}$ gives a reasonable approximation for the actual intersoliton separation but as we proceed to the comparison of the *shapes* of the numerical and variational solution, the agreement deteriorates. The approximation could be improved by decoupling the solitons' amplitudes from their widths and adding the chirp variable; however, even taken in its present form the variational description allows us to draw several principal conclusions. First, the phase variation is

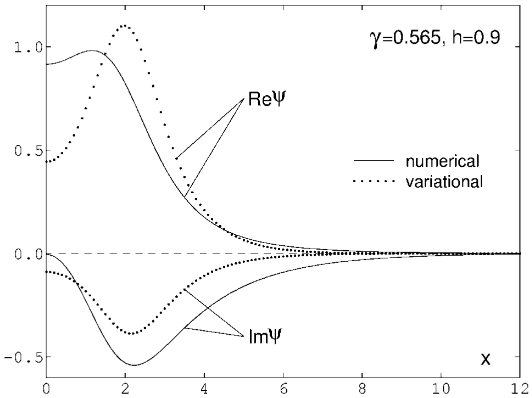

FIG. 1. The variational ansatz (2) with stationary values $\theta = \theta_+$, $z = \tilde{z}$, $A = \tilde{A}$, and $p = \tilde{p}$ (dotted) and the numerically obtained complex $\Psi_{(++)}$ (solid curves). All configurations being symmetric, we show them only for $x > 0$.

an essential ingredient of the complex formation mechanism. Had we not included a nonzero p, Eq. (5) would have given us $\partial_z H_{\rm int}^{(0)} = 0$ whose only root is $z = 0$. A related observation concerns the interpretation suggested in the undamped case [7] where the time-dependent version of Eq. (9), $\dot{z} = p\{\cdots\}$, was used to eliminate p and reduce the finite-dimensional dynamics to an equation for a center-of-mass particle in a potential field $H_{\rm int}^{(0)}(z)$. In this approach the stationary bound state of two solitons would correspond to the particle sitting at the minimum of $H_{\rm int}^{(0)}(z)$, with the momentum $p = 0$. (The problem however is that no such nontrivial minima exist.) On the contrary, our stationary bound state arises when the contents of $\{\cdots\}$ in (9) vanish; this corresponds to the particle with infinite mass and $p \neq 0$. Therefore the formation of complexes in the parametrically driven damped nonlinear Schrödinger equation *cannot* be explained by the two-particle mechanism [2–4,7,21] where one soliton is captured in a potential well formed by its mate. Finally, the variable amplitude A is another essential ingredient. As one can check, if no provision were made for the variation of A, the resulting equations for z and p would have stationary points only for large $h > \sqrt{1.726 + \gamma^2}$. In this region all localized solutions are unstable against radiation waves [16].

Numerical solutions.—We used a predictor-corrector continuation algorithm with a fourth-order Newtonian solver to obtain stationary solutions of Eq. (1). As a bifurcation measure we adopted the energy functional

$$E = \mathrm{Re}\int \{|\Psi_x|^2 + |\Psi|^2 - |\Psi|^4 + h\Psi^2\}\,dx\,, \quad (13)$$

which is conserved when $\gamma = 0$. Our findings are summarized in Fig. 2 where we have also included information on the stability of solutions. This was studied by computing, numerically, eigenvalues of the linearized problem

$$\mathcal{H}y = \mu\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} y, \qquad \mathcal{H} = -\partial_x^2 + \begin{pmatrix} A_+^2 - 6u^2 - 2v^2 & \gamma - 4uv \\ \gamma - 4uv & A_-^2 - 6v^2 - 2u^2 \end{pmatrix}, \quad (14)$$

where $u + iv \equiv \Psi(x)e^{i\theta_+}$; $(\delta u, \delta v)^T = e^{\lambda t}y(x)$, $\lambda = \mu - \gamma$. The phase variable $\chi(x) = -\arg\Psi$ turns out to be useful in the identification of different complexes. For example, the phase of the solution identified as $\Psi_{(+-+)}$ (i.e., a symmetric association of two Ψ_+'s and one Ψ_- in between) is close to θ_- around the central soliton and to θ_+ around the two side ones (Fig. 3). The separation of the Ψ_+ and Ψ_- constituents in this complex is large even for small h ($x_0 \sim 30$ for $h \sim \gamma$). (All numbers are for $\gamma = 0.565$). As $h \to h_c = \sqrt{1 + \gamma^2}$ and the width of the central soliton increases ($1/A_- \to \infty$), the intersoliton separation grows to infinity. If we continue to the right along another branch, $\Psi_{(-+-)}$, the separation decreases from $x_0 \sim 30$ at $h \sim \gamma$ to $x_0 \sim 10$ near the turning point $h = 0.86742$. Turning left and upwards, the separation keeps on decreasing, the central soliton gradually dies out and the complex is made into $\Psi_{(--)}$. After one more turning point at $h = 0.83504$, as we continue to the right, the amplitudes of the constituent solitons start to grow and the complex gradually transforms into $\Psi_{(++)}$. It is interesting to note here that the asymptotic phase of the solution remains equal to θ_- and not θ_+ as could have been expected of a complex of two Ψ_+ solitons (Fig. 3). At $h = 0.9435$ the complex undergoes an "inverse" Hopf bifurcation where a pair of unstable complex-conjugate eigenvalues crosses from $\mathrm{Re}\lambda > 0$ to $\mathrm{Re}\lambda < 0$ half plane. The remaining portion of the $\Psi_{(++)}$ branch (thick line in Fig. 2) represents the only stable bound state in the system; all other complexes were found to be unstable. As h is increased, the intersoliton separation grows but remains finite all the way up to $h = h_c$.

Some insight into the structure of stationary complexes can be gained by noting the law of the variation of the "area" integral, $N = \int |\Psi|^2\,dx$,

$$\dot{N} = 2h\int \rho\{\sin(2\chi) - \sin(2\theta_\pm)\}\,dx\,. \quad (15)$$

Here $\Psi = \sqrt{\rho}\,e^{-i\chi}$. For stationary complexes the integral in the right-hand side has to vanish. This can be easily achieved when solitons bind at a very large separation, as in $\Psi_{(+-+)}$. In this case the variation of χ should mainly be confined to regions where ρ is almost zero (Fig. 3). The resulting contribution to the integral (15) can be offset by small deviations of χ from $\theta_\pm$ around the centers of the solitons, and indeed, a closer inspection reveals that $\sin(2\chi) - \sin(2\theta_\pm)$ assumes small negative values around the core of each soliton bound in $\Psi_{(+-+)}$.

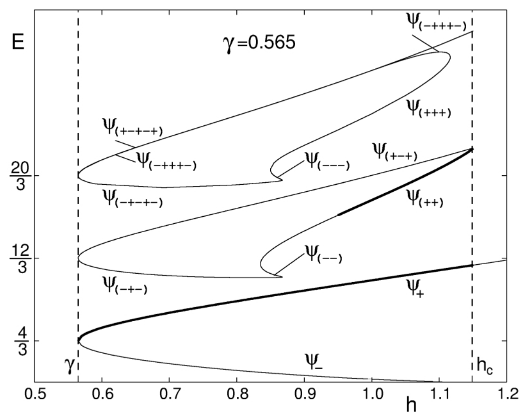

FIG. 2. Existence and stability diagram of one-, two-, and three-soliton stationary solutions. Thick and thin lines depict stable and unstable branches, respectively. The boundary conditions are $\Psi(\pm L) = 0$ where L was typically equal to 100. As $h \rightarrow h_c$, L had to be increased up to 500. Eigenfunctions of (14) were sought for as Fourier expansions over 1000 modes.

Formation mechanism.—As we have mentioned, the binding mechanism is more involved here than just a balance of repulsion and attraction between the two solitons. Details are yet to be elucidated in numerical simulations of the time-dependent equation (1) while here we shall emphasize only its main ingredients. First of all, noting that the amplitude of each soliton is given by Eq. (8), one can check that the area of the configuration (2) is a monotonically growing function of z. Consequently oscillations of the separation between the two solitons are completely characterized by oscillations of N. The dynamics of the latter is described by Eq. (15) where the right-hand side is very sensitive to variations of the phase (in particular, to variations of our p variable). If the complex is in its stage of expansion, at a certain moment of time the phase $\chi(x,t)$ will pass through a configuration rendering the integral in (15) zero. The expansion will then switch to contraction—until the phase is again such that $\dot{N} = 0$. In the stable region of h and γ the oscillations will settle to the stationary complex $\Psi_{(++)}$.

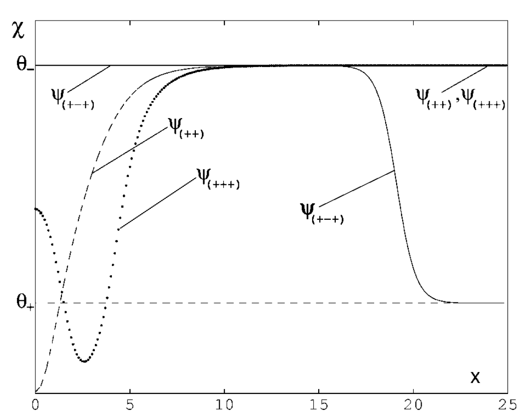

FIG. 3. The phase $\chi(x)$ of the complex $\Psi_{(+-+)}$ (solid line). The Ψ_+ solitons in this complex are centered at $x_0 \approx \pm 37$ while the variation of χ is confined mainly to $17 < |x| < 21$. Also shown is the phase of $\Psi_{(++)}$ (dashed) and $\Psi_{(+++)}$ (dots).

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