Exact Calculation of the Radiatively Induced Lorentz and *CPT* Violation in QED

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Radiative corrections arising from the axial coupling of charged fermions to a constant vector b_μ can induce a Lorentz- and *CPT*-violating Chern-Simons term in the QED action. We calculate the exact one-loop correction to this term keeping the full b_μ dependence, and show that in the physically interesting cases it coincides with the lowest-order result. The effect of regularization and renormalization and the implications of the result are briefly discussed.

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The possible breaking of *CPT* and Lorentz invariance due to nonconventional physics has been recently addressed by constructing extensions of the standard model that include tiny noninvariant renormalizable terms (see Refs. [1–5], and references therein; the possibility of dynamical Lorentz symmetry breaking has been considered in Ref. [6]). In particular, it is interesting to consider the QED sector of such extensions. We shall be concerned here with only the Lorentz-violating *CPT*-odd terms, which for a single charged (Dirac) fermion read

$$S^{CPT} = \int d^4x\Big[-a_\mu\bar\psi\gamma^\mu\psi - b_\mu\bar\psi\gamma^\mu\gamma_5\psi + \frac{1}{2}k^\mu\epsilon_{\mu\nu\rho\sigma}A^\nu F^{\rho\sigma}\Big]. \quad (1)$$

Stringent experimental bounds can be put on the pure-photon *CPT*-violating term [7], which is of the Chern-Simons form [8] (a disputed claim exists, however, for a nonzero $\vec{k}$ [9,10]). Moreover, this term introduces tachyonic modes in the photon spectrum and, for a timelike k_μ, a vacuum instability [7,11]. Hence, both experiment and theory suggest that k_μ should vanish (at least in the timelike case). A natural question is then whether a nonzero k_μ can be induced by radiative corrections involving Lorentz and *CPT*-violating couplings in other sectors of the total low-energy theory. In that case, the tight bounds on k_μ would also constrain these sectors. In the QED extension such corrections can arise only from the axial-vector term, with coupling b_μ.

Several authors have tried to answer this question. All calculations have been performed to one loop and at leading order in b_μ, and have rendered a finite result. However, despite some early claims of definite values for the induced k_μ [3,12], it seems quite clear now that the result is ambiguous [2,13–15], i.e., depends on the details of the high-energy theory [16]. It is our purpose here to calculate the one-loop corrections to all orders in the coupling b_μ and discuss the relevant issues in the light of the exact result. After this work had been completed, we learned that Chung had carried out the same calculation (for $b^2 < m^2$), with the same result [17].

The relevant quantity is the parity-odd part of the vacuum polarization, which must be of the form

$$\Pi^{\mu\nu}_{\rm odd}(p) = \epsilon^{\mu\nu\alpha\beta}b_\alpha p_\beta K(p,b,m), \quad (2)$$

where p_μ is the external momentum and the function K is a scalar. The contribution to the induced Chern-Simons term in the effective action is given by

$$(\Delta k)^\mu = -\frac{1}{2}b^\mu K(0,b,m), \quad (3)$$

and must be a function of b^2/m^2. To one-loop, the only contributing diagram coincides with the standard one-loop vacuum-polarization but with the usual fermion propagator replaced by the b_μ-exact propagator

$$S_b(k) = \frac{i}{\not k - m - \not b\gamma_5}. \quad (4)$$

We use a Hermitian γ_5 with ${\rm tr}\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5\} = 4i\epsilon^{\mu\nu\rho\sigma}$ and $g_{\mu\nu} = {\rm diag}(1,-1,-1,-1)$. In order to keep the full dependence on b_μ we must rationalize the propagator. We find

$$S_b(k) = i\frac{(\not k + m - \not b\gamma_5)(k^2 - m^2 - b^2 + [\not k, \not b]\gamma_5)}{(k^2 - m^2 - b^2)^2 + 4[k^2b^2 - (k\cdot b)^2]}, \quad (5)$$

which agrees with the expression given in Ref. [1]. As discussed there, this propagator has four poles that occur at real values of k_0 [18]. The one-loop vacuum polarization reads

$$\Pi^{\mu\nu}(p) = \int\frac{d^4k}{(2\pi)^4}{\rm tr}\{\gamma^\mu S_b(k)\gamma^\nu S_b(k-p)\}. \quad (6)$$

This integral is linearly divergent. Equation (5) allows us to compute the trace in the numerator, which for the odd terms in b_μ reduces to $\epsilon^{\mu\nu\alpha\beta}p_\beta F_\alpha(k,p,b,m)$, with

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$$
\begin{aligned}
F_\alpha(k,p,b,m) &= b_\alpha F_\alpha^{(1)}(k,p,b,m) + k_\alpha F_\alpha^{(2)}(k,p,b,m) \\
&= -4i\{b_\alpha[2m^2(k^2 - m^2 - b^2) + (k^2 + m^2 + b^2)\big((k-p)^2 - m^2 + b^2\big) + 4k\cdot b(k-p)\cdot b] \\
&\quad - 2k_\alpha[(k^2 - m^2 + b^2)(k-p)\cdot b + \big((k-p)^2 - m^2 + b^2\big)k\cdot b]\}\,. \qquad (7)
\end{aligned}
$$

The linearly divergent term has disappeared, leaving an integral which is just logarithmically divergent by power counting. Since we are interested in only $K(p,b,m)$ for $p_\mu = 0$ and, luckily, no term with $\epsilon^{\mu\nu\alpha\beta}b_\alpha k_\beta$ appears, we can simplify the calculation by setting $p_\mu = 0$ in $F_\alpha(k,p,b,m)$ and in the denominator of the integral. We are then left with two integrals which depend on only b_μ and m. The first one is already of the form $b_\alpha K_1(0,b,m)$. The second integral must also give a result of the form $b_\alpha K_2(0,b,m)$, and $K_2(0,b,m)$ can be calculated multiplying the integral by b_α and dividing by b^2 (as long as b_μ is not lightlike). In this way, we arrive at the following expression:

$$
\begin{aligned}
K(0,b,m) &= -4i\int \frac{d^4k}{(2\pi)^4}\,\frac{1}{[(k^2 - m^2 - b^2)^2 + 4\big(k^2b^2 - (k\cdot b)^2\big)]^2} \\
&\quad \times \left\{[2m^2(k^2 - m^2 - b^2) + (k^2 + b^2)^2 + 4(k\cdot b)^2 - m^4] - \frac{1}{b^2}[4(k\cdot b)^2(k^2 - m^2 + b^2)]\right\}. \qquad (8)
\end{aligned}
$$

In order to calculate this integral, we go to Euclidean space via a Wick rotation of k_0 and then perform an analytic continuation to $b_E = (ib_0, \vec{b})$:

$$
\begin{aligned}
K(0,b_E,m) &= -4\int \frac{d^4k_E}{(2\pi)^4}\,\frac{1}{[(k_E^2 + m^2 - b_E^2)^2 + 4\big(k_E^2b_E^2 - (k_E\cdot b_E)^2\big)]^2} \\
&\quad \times \left\{[2m^2(k_E^2 + m^2 - b_E^2) - (k_E^2 + b_E^2)^2 - 4(k_E\cdot b_E)^2 + m^4] + \frac{1}{b_E^2}[4(k_E\cdot b_E)^2(k_E^2 + m^2 + b_E^2)]\right\}. \qquad (9)
\end{aligned}
$$

Here the scalar product is Euclidean. One can directly see at this stage that the result must be finite. Indeed, for very large $|k_E|$ the leading term in the integrand has the form

$$
\frac{k_E^2 - 4(k_E\cdot b_E)^2/b_E^2}{k_E^6}\,, \qquad (10)
$$

which gives a vanishing result if the integral is done symmetrically. The other terms are power-counting finite. This also shows an ambiguity in the induced term: the integral of the expression (10) is regularization dependent. In (four-dimensional) spherical coordinates Eq. (9) reads

$$
\begin{aligned}
K(0,b_E,m) &= -\frac{1}{\pi^3}\int_0^\infty d|k_E|\,|k_E|^3\int_0^\pi d\theta\,\sin^2\theta\,\frac{1}{[(|k_E|^2 + m^2 - b_E^2)^2 + 4|k_E|^2b_E^2\sin^2\theta]^2} \\
&\quad \times \{[2m^2(|k_E|^2 + m^2 - b_E^2) - (|k_E|^2 + b_E^2)^2 - 4|k_E|^2b_E^2\cos^2\theta + m^4] \\
&\quad + [4|k_E|^2(|k_E|^2 + m^2 + b_E^2)\cos^2\theta]\}\,. \qquad (11)
\end{aligned}
$$

Doing first the angular integral we find

$$
\begin{aligned}
K(0,b_E,m) &= \frac{1}{4\pi^2b_E^4}\int_0^\infty d|k_E|\,|k_E|\left\{(k^2 + m^2) - \mathrm{sgn}(|k_E|^2 + m^2 - b_E^2)\right. \\
&\quad \left.\times \frac{(|k_E|^2 + m^2)^4 + 3b_E^2(|k_E|^2 + m^2)^2(|k_E|^2 + b_E^2 - m^2) + b_E^6(|k_E|^2 - m^2)}{[4b_E^2|k_E|^2 + (|k_E|^2 + m^2 - b_E^2)^2]^{3/2}}\right\}. \qquad (12)
\end{aligned}
$$

This integral is well behaved for large $|k_E|$. Note the appearance of the sign function. For $m \neq 0$, the final result is (going back to the Minkowskian b_μ)

$$
(\Delta k)^\mu = \frac{3}{16\pi^2}\,b^\mu, \quad \text{if } -b^2 \le m^2; \qquad (13)
$$

$$
(\Delta k)^\mu = \left(\frac{3}{16\pi^2} - \frac{1}{4\pi^2}\sqrt{1 - \frac{m^2}{|b^2|}}\right)b^\mu, \quad \text{if } -b^2 > m^2. \qquad (14)
$$

For any timelike b_μ and for a spacelike b_μ with $|b^2| < m^2$, Eq. (13) is the relevant one. Surprisingly enough, in these cases our calculation to all orders in b_μ gives the same result as the one obtained in the b_μ-linear approximation of Ref. [15]. Obviously, perturbation theory about $b_\mu = 0$ does not detect the different behavior we have found for $-b^2 > m^2$. On the other hand, continuity in b^2 implies that the lightlike case, $b^2 = 0$, is also given by Eq. (13). In fact, dimensional analysis shows that the b_μ-linear approximation is exact for vanishing b^2. Note also in passing that for the fine-tuned value $b^2 = -\frac{16}{7}m^2$, a vanishing $(\Delta k)^\mu$ is obtained.

If the fermion is massless, $m = 0$, we find $(\Delta k)^\mu = -\frac{1}{16\pi^2}b^\mu$ for any kind of b_μ. In this simple case, there are at least three other possible ways of calculating the induced term, which give the same answer:

(1) A simpler b_μ-exact propagator can be obtained for $m = 0$:

$$S_b(k) = i\frac{(\not{k} + \not{b}\gamma_5)(k^2 + b^2 - 2k \cdot b\gamma_5)}{(k + b)^2(k - b)^2}. \qquad (15)$$

The calculation can then be simplified using Feynman parameters and the result $(\Delta k)^\mu = -\frac{1}{16\pi^2}b^\mu$ is found for any b_μ.

(2) Perturbatively, one can perform just a b_μ-linear calculation, since in the massless case the higher-order terms vanish for dimensional reasons. From Eq. (14) in Ref. [15], the same $(\Delta k)^\mu$ results. The only contribution comes from the surface term in Eq. (10) of Jackiw-Kostelecký's calculation.

(3) A confirmation of the same result, nonperturbative in b_μ and in the fine-structure constant, is provided in Ref. [13] (following a suggestion by D. Colladay): an anomalous chiral redefinition of the fermion fields allows one to get rid of the coupling to b_μ, so that the contribution (to all orders) to $K(0, b, 0)$ comes from the corresponding Fujikawa Jacobian. Up to the unavoidable ambiguity (which in this method comes from the definition of the current operator), $-\frac{1}{16\pi^2}b^\mu$ is obtained again.

An infrared regularization can spoil this result. For instance, giving the fermion a small mass obviously shifts it back to $\frac{3}{16\pi^2}b^\mu$ in the timelike-b_μ case. At any rate this is just a formal discussion, since there are no massless electromagnetically charged fermions in nature.

We have also calculated the integral in Eq. (8) directly in Minkowski space, using first the residues method to perform the integration on k_0, and integrating on $\vec{k}$ afterwards. Although the result differs by a constant, because k_0 and $\vec{k}$ are not treated symmetrically, the same dependence on b^2 and m^2 is found.

It is rather striking that the contributions to $(\Delta k)^\mu$ of diagrams with more than one insertion of $\not{b}\gamma_5$ vanish. We have explicitly checked that this is indeed the case at order $b_\mu b^2$. At this and higher orders, all integrals are finite by power counting, and hence unambigous. Coleman has observed that the vanishing of these higher-order contributions can be explained by his argument with Glashow in Ref. [3], which can be applied to finite diagrams with insertions of $\not{b}\gamma_5$ [19]. Consider a two-photon amplitude with $n > 1$ insertions of $\not{b}\gamma_5$. The idea is to let each of the two photons carry different momenta, p and q (the insertions carry nonzero momentum then). Gauge invariance implies transversality in each of the photons. Differentiating each of the transversality conditions with respect to the corresponding momentum, one learns that the amplitude is $O(pq)$. It follows that when ones goes to equal momenta, $q = p$, the amplitude is $O(p^2)$. Since the Chern-Simons term in the effective action is $O(p)$, one concludes that this amplitude does not contribute to $(\Delta k)^\mu$. In Ref. [3], this argument is proved to be valid to any order in the fine-structure constant. Note, however, that it does not apply to diagrams with just one insertion, due to the presence of triangular anomalies [15].

Let us discuss now how regularization and renormalization affect the result. This is an important point because the complete $S^{\mathrm{QED}} + S^{CPT}$ theory is not finite and requires renormalization (and, furthermore, renormalization is also relevant in a finite theory [20,21]). The *exact* decomposition $S_b(k) = S(k) - iS_b(k)\not{b}\gamma_5 S(k)$ performed in Ref. [15], shows that the ambiguities can come from only the lowest-order piece, the rest being finite by power counting. In our calculation, the result can be changed by any regularization that destroys either the spherical symmetry (in four-dimensional Euclidean space) of the high-energy behavior or the steps we followed to arrive at Eq. (9). In general, different regularizations (or subtractions) will render different results, even if they preserve gauge invariance. This is apparent in differential renormalization, which makes the ambiguity explicit [14]. As a matter of fact, independently of how one regulates and subtracts the divergent integrals, one always has the freedom to add any (renormalizable) finite counterterm that is allowed by the relevant symmetries of the theory [22]. This is also true when the radiative corrections to that term are finite [21]. In the present case, this means that the induced $(\Delta k)^\mu$ can have any value, for it is not protected by any symmetry [15] (except CPT and Lorentz invariance, but we just broke them). The conclusion of our study is the following: if the regulator and the subtraction rule are mass independent, a mass-independent result will be obtained to all orders in b_μ in the physically relevant cases, as the CPT- and Lorentz-violating terms coefficients are much smaller than the mass of any electromagnetically charged fermion and Eq. (13) provides the induced term in this situation.

In Ref. [2] an interesting discussion was made regarding the possible vanishing of the induced Chern-Simons term due to an anomaly-cancellation mechanism in the high-energy theory (of course, we consider now several fermion species). Essentially, the argument goes as follows: From the point of view of a more fundamental theory, the diagrams with one $\not{b}\gamma_5$ insertion (at one loop) can be viewed as the corresponding triangular diagrams with the same photon legs and a third leg involving a coupling to an axial vector, in the limit in which there is zero momentum transfer to the axial-vector leg and the latter is replaced with a vacuum expectation value. The condition for the cancellation of the anomalies occurring in these diagrams then implies that the induced term also cancels. This argument requires that the term induced by different fermions be the same. This is true if the induced term contains no mass dependence and, besides, a universal and mass-independent renormalization prescription is adopted for all the contributing diagrams (such a prescription can be justified again by a similar argument). Our result shows that the first requirement, which is trivial at the b_μ-linear order, holds to all orders in b_μ. Invoking the Adler-Bardeen theorem, the argument for the vanishing of the induced term was also generalized in Ref. [2] to higher loops. Again, this is valid only if no mass dependence is introduced by higher-order corrections. That this is the case follows from the combination of Coleman-Glashow's argument and Adler-Bardeen theorem. To summarize, in the context of Ref. [2], the cancellation of anomalies in the fundamental theory imply the vanishing of the induced term to any order in b_μ and in the fine-structure constant, if a mass-independent scheme is used.

We have until now neglected the possible influence of the a_μ term in Eq. (1) but, in principle, corrections of order $a^2 b_\mu$ and higher could appear. Actually, one can include the effect of a_μ to all orders by considering the corresponding a_μ- and b_μ-exact propagator. This propagator is just the one in Eq. (5) but with k_μ substituted by $k_\mu - a_\mu$. The vector a_μ behaves then like an external momentum which appears in all the propagators of the loop. Since there are no derivative couplings, a shift in the loop momentum $k_\mu \rightarrow k_\mu + a_\mu$ can completely eliminate a_μ, so the result is not affected. This shift is also subjected to regularization ambiguities. As a matter of fact, the term proportional to a_μ can be eliminated from the action $S^{\mathrm{QED}} + S^{CPT}$ by a field redefinition of the form $\psi = \exp(-ia \cdot x)\chi$ [1]. The effect of other sectors (like the CPT-even Lorentz-violating extension of QED considered in Ref. [2]) can also be studied with these techniques, i.e., incorporating the corresponding fermion bilinears into the exact propagator. This is beyond the scope of the present work.

Let us finally stress that even if the radiative corrections to the Chern-Simons term cancel for some reason, it is still possible to add a finite counterterm and get a nonzero $(\Delta k)^\mu$. This is a sign of the fact that we have no right to put $k^\mu = 0$ at tree level, unless there is some symmetry in the high-energy theory that imposes this value. Of course, it is comparison with experiment that tells us to set $k^\mu + (\Delta k)^\mu = 0$ but, in the absence of a proper symmetry, we are facing a problem of naturalness. Nevertheless, the anomaly-cancellation argument shows that this problem does not come from other sectors of the low-energy theory, and, from a practical point of view, allows one to put $k^\mu = 0$ in the tree-level action, as long as a consistent mass-independent scheme is used in the calculations.

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