$1/f$ Noise in Bak-Tang-Wiesenfeld Models on Narrow Stripes

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We report our findings of a $1/f$ power spectrum for the total amount of sand in directed and undirected Bak-Tang-Wiesenfeld models confined to narrow stripes and driven locally. The underlying mechanism for the $1/f$ noise in these systems is an exponentially long configuration memory giving rise to a very broad distribution of time scales. Both models are solved analytically with the help of an operator algebra to explicitly show the appearance of the long configuration memory.

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The ubiquitous $1/f$ noise fascinated physicists for generations [1]. This phenomenon usually indicates a broad distribution of time scales in the system. One simple scenario of having an exponentially broad distribution of time scales is through an exponential dependence of local characteristic frequency on some parameter [2], e.g., thermal activation events in equilibrium systems over a sufficiently broad and flat distribution of energy barriers [3]. To account for broad scale distributions in space and time in interacting dynamical systems, Bak, Tang, and Wiesenfeld (BTW) introduced the notion of self-organized criticality (SOC) [4]. Although the original BTW "sandpile" model did not exhibit the $1/f$ power spectrum *per se* [5], variants of the model indeed show $1/f$-like noise [6]. In this Letter, we report the observation of $1/f$ noise for directed and undirected BTW models confined to narrow stripes (quasi-one-dimensional geometries). In these models, sand flows in the long direction, with periodic or closed boundary conditions in the other direction. The system is driven locally by randomly adding sand to a unique set of sites that have the same coordinate along the long axis. The total amount of sand in the sandpile as a function of time exhibits a clean $1/f$ power spectrum with an exponentially small lower cutoff. Surprisingly, the outcome of these interacting systems falls into the above mentioned simple scenario—the dynamics organizes the system into a state with exponentially long configuration memory.

Let us first consider the directed model [7]. An integer variable $z(x,y)$ is assigned on each site (x,y) of a two-dimensional lattice of size $L_x \times L_y$ ($1 \le x \le L_x$, $1 \le y \le L_y$). Throughout the paper, we refer to $z(x,y)$ as the number of grains of sand (or height) at the site (x,y). The dynamics consists of the following steps: (i) Add a grain of sand to a randomly selected site in the *first* column, $(1,y)$: $z(1,y) \to z(1,y) + 1$; (ii) if as a result of the process the height $z(x,y)$ exceeds a critical value $z_c = 2$, the site topples and three grains of sand are redistributed from this site to three of its nearest neighbors up, down, and to the right; (iii) repeat step (ii) until all sites are stable, i.e., $z(x,y) \le 2$ everywhere. This chain reaction of updates is referred to as an avalanche; and (iv) when the avalanche is over, measure the total amount of sand in the system $\mathcal{Z}(t) = \sum z(x,y)$. Then go to step (i). Notice that the flow of sand is *directed* to the right along the x axis. Also note the separation of time scales—the duration of individual avalanches is taken to be much faster than the unit time interval defined by the addition of sand grains. The boundary condition in the x direction is always set to open: $z(L_x + 1, y) = 0$. While in the y direction we either choose the periodic boundary condition (which we refer to as model 1) or the closed boundary condition. In the latter case we restrict ourselves to $L_y = 2$ and refer to it as model 1A. In model 1A, we set $z_c = 1$ and the redistribution rule (ii) prescribes to move two grains of sand from the toppling site: one to the right along the x direction and the other to its nearest neighbor (up or down) in the y direction.

After some transient period the above dynamics brings the system to a stationary state, where the total amount of sand in the system $\mathcal{Z}$ saturates and fluctuates about its average value. At this point we start recording $\mathcal{Z}(t)$ and measure its power spectrum $S(f) = |\hat{\mathcal{Z}}(f)|^2$, where $\hat{\mathcal{Z}}(f)$ is the Fourier transform of $\mathcal{Z}(t)$. In Fig. 1(a), we show the power spectra for models 1 and 1A. Even for small systems one observes a very broad $1/f$ region. In fact, as we will demonstrate later, the lower cutoff of the $1/f$ region falls off with L_x exponentially. Our simulations indicate that as the width of the stripe L_y is increased the $1/f$ region shrinks, as shown in the bottom curve of Fig. 1(a). Direct observations of configurational changes at each time step clearly indicate that the rate of configurational changes at x decreases drastically with increasing x [8], suggesting that there are many time scales and some kind of long memory in the system. To understand this, we proceed with solving model 1A using the group of operators introduced by Dhar [9].

To simplify the notation let us denote the configuration at the pair of sites $(x,1)$ and $(x,2)$ by the column $\binom{z(x,2)}{z(x,1)}$, and let $L_x = L$. Any pair configuration with both

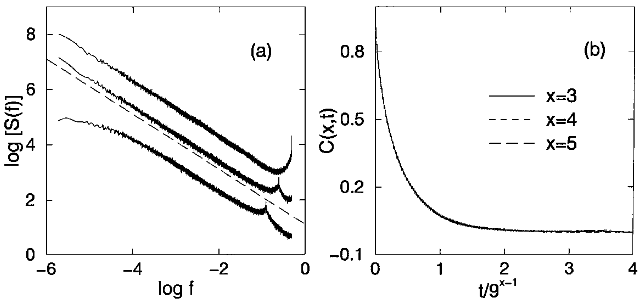

FIG. 1. (a) From top to bottom, power spectra for model 1A with $L_x = 8$, model 1 with $L_x = 8$ and $L_y = 4$, and model 1 with $L_x = 8$ and $L_y = 8$. The second and the third curves are shifted vertically by -1 and -2 decades, respectively, for clarity. The dashed line has the slope -1. (b) Autocorrelation functions $C(x,t)$ at $x = 3$, $x = 4$, and $x = 5$, for model 1A with $L_x = 8$.

$z(x,y) \le z_c = 1$ is stable. However, the recurrent pair configurations, present in the stationary SOC state, are $\binom{0}{1}$, $\binom{1}{0}$, and $\binom{1}{1}$, while $\binom{0}{0}$ is never realized after the transient period. There are no additional restrictions on pair configurations at different columns [7]; the total number of recurrent states is thus 3^L. Let us define operators U_x and D_x acting on recurrent configurations. The action of these operators consists of adding one grain of sand at sites $(x,2)$ and $(x,1)$ correspondingly, and, if necessary, relaxing the resulting configuration according to the avalanche rules. The final stable configuration is the result of the operator acting on the initial configuration. The following operator identities [9] result directly from the avalanche rules

$$U_x^2 = D_x U_{x+1}\,, \tag{1}$$

$$D_x^2 = U_x D_{x+1}\,. \tag{2}$$

Open boundary at $x = L + 1$ corresponds to $U_{L+1} = D_{L+1} = \mathbf{I}$, where $\mathbf{I}$ is the identity operator. From Eqs. (1) and (2) it immediately follows that

$$U_1 D_1 = U_2 D_2 = \cdots = U_L D_L = U_{L+1} D_{L+1} = \mathbf{I}\,. \tag{3}$$

In other words, addition of one grain of sand to both upper and lower sites at any column x triggers a downstream avalanche, in which two grains fall off the right edge, leaving the underlying configuration unchanged. Since $D_x = U_x^{-1}$, Eq. (1) can be rewritten as $U_x^3 = U_{x+1}$. Therefore, $U_1^{3^L} = U_2^{3^{L-1}} = U_L^3 = U_{L+1}$, and $U_1^{3^L} = \mathbf{I}$. Repeated application of U_1 3^L times makes the system visit every one of the 3^L recurrent states exactly once and return to its original configuration. If U_1 and D_1 are applied in random order (that is how we drive the system), since $U_1^m D_1^n = U_1^{m-n}$, the average time required to visit all 3^L states is given by $\sqrt{T} = 3^L$, or $T = 9^L$.

Now let us study the "microscopic" details of how operators U_x and D_x change the configurations of the sandpile. Note that $U_x\binom{1}{0} = \binom{0}{1}U_{x+1}$, $U_x\binom{0}{1} = \binom{1}{1}$, and $U_x\binom{1}{1} = \binom{1}{0}\mathbf{I}$. It is clear that the only cases where the operator U_x or D_x can "propagate" to the right and change the state of the next neighbor are $U_x\binom{1}{0} = \binom{0}{1}U_{x+1}$ and $D_x\binom{0}{1} = \binom{1}{0}D_{x+1}$. So, in order for the operator U_1 to be able to change the configuration of a pair at x, all $x - 1$ pairs to the left of this pair must be in the state $\binom{1}{0}$. Thus, the probability that a pair at x changes its configuration as a result of the addition of a grain of sand at the left end of the system is $1/3^{x-1}$. If the system is driven by a random sequence of U_1 and D_1, on average it will take $\tau_x \sim (3^{x-1})^2 = 9^{x-1}$ grains of sand to change the configuration of a pair at x.

This exponential increase of the characteristic time τ_x with x is manifested in the local autocorrelation functions $C(x,t) = [\langle Z(x,0)Z(x,t)\rangle - \langle Z(x,0)\rangle^2]/[\langle Z(x,0)^2\rangle - \langle Z(x,0)\rangle^2]$, where $Z(x,t) = z(x,1) + z(x,2)$ is the number of grains at column x and at time t. In Fig. 1(b) we plot $C(x,t)$ vs $t/9^{x-1}$ for several x's. One sees that $C(x,t) = F(t/9^{x-1})$. This form of the local autocorrelation function implies a scaling form for the local power spectrum: $S_{\mathrm{loc}}(f,x) = [1/f_{\mathrm{char}}(x)]S\big(f/f_{\mathrm{char}}(x)\big)$ with $f_{\mathrm{char}}(x) = f_0 \exp(-x \ln 9)$, where $S_{\mathrm{loc}}(f,x)$ is the power spectrum of $Z(x,t)$. Note that if U_x or D_x propagates through a column it leaves the number of grains on that column $Z(x)$ unchanged. It follows that the addition of a grain at the left end of the system can change at most the number of grains at one column. If we assume that the local events of changing $Z(x)$ are independent for different x's (or the correlations are not too strong), which is a reasonable approximation when we drive model 1A with a random sequence of U_1's and D_1's, then the global power spectrum of the total number of grains in the system is the superposition of the local power spectra. The exponential fall off of the local characteristic frequencies of configuration changes would give rise to a global $1/f$ power spectrum. That is [2,3]

$$\begin{aligned} S_g(f) &= \int_0^L S_{\mathrm{loc}}(f,x)\,dx \\ &= \int_0^L [1/f_{\mathrm{char}}(x)]S\big(f/f_{\mathrm{char}}(x)\big)\,dx \\ &= \int_0^L \exp(\lambda x) S\big(f\exp(\lambda x)/f_0\big)\,dx/f_0 \\ &= (1/f)\int_{f/f_0}^{f\exp(\lambda L)/f_0} dy S(y)/\lambda\,. \end{aligned}$$

The lower cutoff of the $1/f$ region is $f_c \sim f_0 \exp(-\lambda L)$, which for the top curve in Fig. 1 ($\lambda = \ln 9$ and $L = 8$) is of the order of 10^{-7}.

We now turn our attention to the undirected model on a stripe $L_x \times L_y$. In this model an unstable site with $z(x,y) > z_c = 3$ redistributes one grain of sand to each of its four neighbors. In our simulations at each time step we randomly select a site on the central column [$x = (L_x + 1)/2$, or, if L_x is an even number we randomly select one of the $2L_y$ sites on the two central columns]

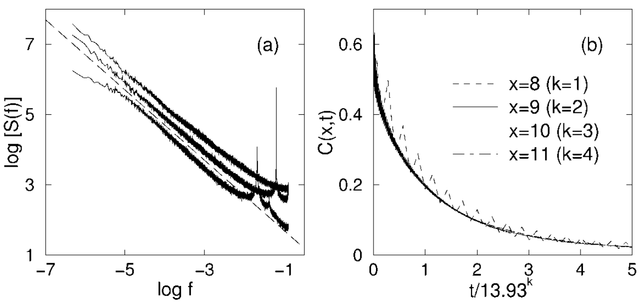

FIG. 2. (a) Power spectra for the center-driven BTW models. From top to bottom: model 2A with $L_x = 16$, model 2 with $L_x = 17$ and $L_y = 8$, and model 2 with $L_x = 17$ and $L_y = 24$. The dashed line has the slope -1. (b) Autocorrelation functions $C(x,t)$ for model 2A with $L_x = 12$, at $x = 8, 9, 10, 11$, which are $k = 1, 2, 3, 4$ distance away from the driving pairs $x = 6$ and $x = 7$.

and add one grain of sand to that site. We choose to have open boundaries at $x = 0$ and $x = L_x + 1$. Again, we study two versions of the model—the first with periodic boundary condition along the y direction (model 2) and the second defined on an $L \times 2$ stripe with closed boundary condition in the y direction (model 2A). In model 2A $z_c = 2$ and a site with $z(x,y) > 2$ moves one grain of sand to each of its three neighbors. In Fig. 2(a), we show the power spectra of the total amount of sand in models 2 and 2A. Similar to the case of the directed model 1, in model 2 the $1/f$ region shrinks when L_y is increased. The dynamics of the undirected models is apparently more complex than that of the directed ones. However, much of the apparent complexity is due to the motion of "troughs" [8]—columns in which *all* $z \leq z_c - 1$, so that avalanches cannot propagate beyond them [10]. We first concentrate on understanding the trough dynamics.

Let us focus on model 2A. The operator relations (1) and (2) now become

$$U_x^3 = D_x U_{x+1} U_{x-1}\,, \tag{4}$$

$$D_x^3 = U_x D_{x+1} D_{x-1}\,. \tag{5}$$

The open boundaries at two ends imply $U_0 = D_0 = U_{L+1} = D_{L+1} = \mathbf{I}$. Let us define the operator $O_x = U_x D_x$. In model 1A we have shown that $O_x = \mathbf{I}$ for every x. This is not so in the undirected model. However, in this model the operators O_x form a simple small subgroup of all operators in the system. From Eqs. (4) and (5) it follows that $O_x^2 = O_{x+1} O_{x-1}$. Using this operator identity repeatedly one gets $O_x^2 = O_{x+1} O_{x-1} = O_{x+2} O_{x-2} = \cdots = O_{x+n} O_{x-n}$. And in general,

$$O_x O_{x'} = O_{(x+x') \bmod (L+1)}\,. \tag{6}$$

In other words, operators $O_0\ (= \mathbf{I}), O_1, O_2, \ldots, O_L$ form a cyclic subgroup of $L + 1$ elements. To understand the physical nature of this subgroup let us take a closer look at the set of recurrent configurations in model 2A. If a stable subconfiguration at a subset of sites F does not occur in the recurrent states, it is called a *forbidden subconfiguration* (FSC) [9]. A subconfiguration on F is an FSC if for every site $(x,y) \in F$ $z(x,y)$ is strictly smaller than the number of its neighbors *in the subset* F [9]. It is clear that the pair $\binom{0}{0}$ is an FSC. Let us refer to pairs $\binom{1}{1}$, $\binom{0}{1}$, $\binom{1}{0}$ as *troughs*. It is easy to see that a subconfiguration enclosed by two troughs is an FSC. Thus a recurrent configuration cannot contain more than one trough. Therefore, all SOC states fall into one of the $L + 1$ classes: those with no troughs ($|S\rangle$), and those with a trough in the mth column ($|S_m\rangle$). The action of $O_m |S_m\rangle$ does not produce any topplings but simply fills up the trough. On the other hand, $O_k |S_m\rangle$ ($k \neq m$) creates an avalanche in which two grains of sand fall off the pile. However, this avalanche produces only minor changes in the configuration of the pile. Indeed, since $O_k = O_m O^{-1}_{(m-k) \bmod (L+1)}$ the action of $O_k |S_m\rangle$ fills up the trough at $x = m$ and creates a trough at $x = (m-k) \bmod (L+1)$ [11]. The action of O_k on a state $|S\rangle$ with no troughs results in a system-wide avalanche with four grains of sand falling off the pile. The only configurational change, however, is the creation of a new trough at $L + 1 - k$ (recall that $O_k = O^{-1}_{L+1-k}$). These rules mean that the action of the L operators O_k results only in the motion, creation, and annihilation of the trough, and does not destroy the configuration memory of the system.

Having understood the role of the operators O_k, we separate the trivial trough dynamics from others by defining the equivalence relation of operators. If $A = BO_k$, we say that A is equivalent to B and denote it by $A \cong B$. Thus $U_k \cong D_k^{-1}$. One can rewrite the basic operator identity (4) as $U_k^4 = U_{k-1} U_{k+1} O_k$, or

$$U_k^4 \cong U_{k-1} U_{k+1}\,. \tag{7}$$

Since $U_0 = \mathbf{I}$, Eq. (7) implies $U_1^4 \cong U_2$. Write $U_1^{N(k)} \cong U_{k+1}$. One has $U_1^{4N(k)} \cong U_{k+1}^4 \cong U_k U_{k+2} \cong U_1^{N(k-1)} U_1^{N(k+1)}$, which gives the recursion relation $N(k+1) = 4N(k) - N(k-1)$ with initial conditions $N(0) = 1$, $N(1) = 4$. It is easy to show that $N(k) = [(3 + 2\sqrt{3})(2 + \sqrt{3})^k - (2\sqrt{3} - 3)(2 - \sqrt{3})^k]/6$. In a system of size L one has $U_1^{N(L)} \cong U_{L+1} = \mathbf{I}$. The total number of recurrent states is then $N_{\mathrm{SOC}}^{(2A)} = (L+1)N(L)$. In other words, any recurrent configuration can be obtained from a given one by the action of some power of U_1 and, if necessary, creation, annihilation, or change of the position of the trough achieved by the action of L operators O_k. Asymptotically, only $2 + \sqrt{3} \simeq 3.732$ pair configurations per site are allowed in a recurrent state, compared to 9 stable pair configurations.

Now we are in the position to address the question of long memory in model 2A. Let us restrict ourselves to the operator U_k acting on a state that has no trough. We have $U_k \binom{0}{2} = \binom{1}{2}$; $U_k \binom{1}{2} = \binom{2}{2}$; $U_k \binom{2}{0} = \binom{0}{1} U_{k+1} U_{k-1}$;

$U_k\binom{2}{1} = \binom{2}{0}O_k$; $U_k\binom{2}{2} = \binom{2}{1}O_k$. It seems that the third relation could propagate U_k through a string of $\binom{2}{0}$'s, changing the configurations away from the driving point, similar to the case of directed models. This is not so, because one cannot have consecutive columns of $\binom{2}{0}$ in a recurrent state. In fact it is the last two relations which can cause configuration changes away from the driving point. Naively, according to these relations, the action of U_k on $\binom{2}{1}$ or $\binom{2}{2}$ causes only local changes apart from some trough dynamics. However, this is true only if the local changes $[\binom{2}{1} \to \binom{2}{0}$ or $\binom{2}{2} \to \binom{2}{1}]$ do not result in any FSC. If an FSC does appear as a result of the change, the change in the original configuration will not be restricted to one pair, but instead will propagate throughout the FSC. The FSC responsible for such a propagation is the string $(0 \ 1 \ 1 \ \cdots \ 1 \ 0)$. This FSC can be created by the action of U_k on a column of $\binom{2}{1}$ followed by a string of $\binom{2}{1}$'s ended with $\binom{2}{0}$. Or it can be created by acting U_k on a column $\binom{2}{2}$ that is inside a string of $\binom{2}{1}$'s bounded between two $\binom{2}{0}$'s. In both cases, direct application of the avalanche rules shows that all pairs associated with this FSC and the pairs next to it would be updated. These scenarios require that the starting configuration contains a string of $\binom{2}{1}$'s with $\binom{2}{0}$ at at least one of the ends. Such a string of length x is just one among $N_{\mathrm{SOC}}^{(2A)}(x) \sim (2 + \sqrt{3})^x \simeq 3.732^x$ recurrent states of a string of x columns. That is why the irreversible changes of pairs at distance x from the driving pair are exponentially unlikely. In model 2A, driven by random addition of sand at sites on the central pair(s) the characteristic frequency at distance k from the driving point is given by $f_{\mathrm{char}}(k) \sim 1/[(2 + \sqrt{3})^k]^2 = 13.93^{-k}$ [see Fig. 2(b)].

In spite of the apparent differences between directed and undirected models, the mechanism for a long term memory and the $1/f$ spectrum is the same: (a) Operators $O_x = U_x D_x$ do not produce irreversible changes in the configuration; (b) In order to produce irreversible changes at a distance k from the place of sand addition, all k pairs in between have to be in a unique peculiar configuration (out of $N_{\mathrm{SOC}}(k) \sim A^k$ possible recurrent subconfigurations); (c) Such an exponential dependence of the characteristic frequency on distance leads to the $1/f$ spectrum of the total amount of sand. Note that the observed $1/f$ noise is not related to a power-law distribution of avalanches. In fact in our models these two properties are mutually exclusive. Also note that a $1/f$ spectrum in the total amount of sand would imply a spectrum $\propto f$ for the sand falling off the edge. It is straightforward to generate the case to higher dimensions in which sand flows in one direction with closed or periodic boundaries in other directions. Recently, De Los Rios and Zhang [12] observed a $1/f$ spectrum in a nonconserved sandpilelike model in which a certain fraction of sand is lost in each toppling process. Because of the absence of conservation, avalanches themselves are exponentially unlikely to reach a distant site, giving rise naturally to an exponential distribution of time scales. In contrast, in our model avalanches constantly pass through the system but they produce only small changes of the configuration. A $1/f$ spectrum was also observed previously for a *continuously* boundary-driven BTW model [13]. Its origin was attributed to a (linear) diffusion of $z(x,y)$ with a noisy boundary condition [13,14], which gives a power-law lower cutoff $f_c \sim 1/L_x^2$ for the $1/f$ spectrum—a mechanism very different from ours. Finally, as a possible experimental realization of the model, we suggest the system of superconducting vortices confined to a quasi-1D geometry.

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