Sum Rule of Hall Conductance in a Random Quantum Phase Transition

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The Hall conductance σ_{xy} of two-dimensional lattice electrons in a random potential is investigated. The randomness induces a quantum phase transition where the sum rule of σ_{xy} plays an important role. Using the string (anyon) gauge, numerical study becomes possible in sufficiently weak magnetic fields essential to the floating scenario in the continuum model. Topological objects in the Bloch wave functions (charged vortices) are obtained explicitly. Anomalous plateau transitions ($\Delta\sigma_{xy} = 2, 3, \ldots > 1$) and the trajectory of delocalized states are discussed as well.

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Effects of randomness are crucial in the quantum Hall effect (QHE) [1]. According to the scaling theory of the Anderson localization, all states in two dimensions are localized due to randomness [2]. There are, however, some exceptions. Symmetry effects, which govern universality classes of the Anderson localization, allow the existence of delocalized states in several two-dimensional systems. For example, states at the center of each Landau band become delocalized in the presence of a strong magnetic field. They are not extended in a usual manner, but critical which is associated with multifractal character. They play an essential role in the quantization of the Hall conductance (QHE) and their behavior determines the plateau transition in QHE. The plateau transition occurs as the strength of randomness or magnetic field is varied. It is a typical quantum phase transition and the Hall conductance characterizes each phase.

Nonzero Hall conductance means the existence of delocalized states below the Fermi energy. When the strength of randomness is sufficiently strong, the system is expected to become the Anderson insulator. Then it means disappearance of the delocalized states below the Fermi energy. If we assume that the delocalized states do not disappear discontinuously, they must float upward across the Fermi energy. This is the floating scenario for the delocalized states [3,4]. It is later extended in the discussion of the global phase diagram [5]. The scenario also predicts the selection rule between different integer quantum Hall states. The rule prohibits transitions $\Delta\sigma_{xy} \neq \pm 1$. On the other hand, the breakdown of this selection rule is observed in some experiments [6–8] and numerical simulations [9].

In this paper, based on topological arguments of the Hall conductance [10,11] and numerical study of the lattice model, we try to clarify the points. The plateau transition in QHE is a quantum phase transition where the *sum rule* restricts the transition type. Therefore it is also interesting as a problem of the quantum phase transition. There are several studies on the delocalized states in two-dimensional lattice electrons with uniform magnetic field and random potential [9,12–15]. Here we make clear the topological nature of the Bloch wave functions and the Hall conductance in a sufficiently weak magnetic field regime. It has much to do with the continuum model and may shed some light on the experiments [6–8]. The physical reason of the anomalous plateau transition is stated clearly in our paper.

The Hamiltonian is defined on a two-dimensional square lattice as

$$H = \sum_{\langle l,m\rangle} c_l^\dagger e^{i\theta_{lm}} c_m + \text{H.c.} + \sum_n w_n c_n^\dagger c_n ,$$

where $c_n^\dagger(c_n)$ creates (annihilates) an electron at a site n and $\langle l, m\rangle$ denotes nearest-neighbor sites. The magnetic flux per plaquette, ϕ, is given by $\sum_{\text{plaquette}} \theta_{lm} = 2\pi\phi$, where the summation runs over four links around a plaquette. The last term, $w_n = W f_n$, is the strength of random potential at a site n, and f_n's are uniform random numbers between $[-1/2, 1/2]$. Although the system is infinite, two-dimensional periodicity of $L_x \times L_y$ is imposed on θ_{lm} and w_n (the infinite size limit corresponds to $L_x, L_y \to \infty$). When the randomness strength is weak and the temperature is sufficiently low, the interactions between the electrons may play a dominant role. However, we focus on the situation where it is not important and use this noninteracting model.

When the Fermi energy lies in the lowest jth energy gap, the Hall conductance σ_{xy} is obtained by summing the Chern number C_n below the Fermi energy,

$$\sigma_{xy} = \sum_{n=1}^{j} C_n ,$$
$$C_n = \frac{1}{2\pi i}\int d\boldsymbol{k}\, \hat{z}\cdot(\boldsymbol{\nabla}_k \times \boldsymbol{A}_n),$$
$$\boldsymbol{A}_n = \langle u_n(\boldsymbol{k})|\boldsymbol{\nabla}_k|u_n(\boldsymbol{k})\rangle ,$$

where $|u_n(\boldsymbol{k})\rangle$ is a Bloch wave function of the nth energy band with $L_x L_y$ components, and $u_n^\gamma(\boldsymbol{k})$ is the γth component. The integration is over the Brillouin zone. Arbitrarily choosing the α and βth components of the wave function and focusing on the winding number

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(vorticity or charge) at each zero point (vortex) of the αth component, the expression is rewritten as

$$C_n = \sum_{\ell} N_{n\ell} ,$$
$$N_{n\ell} = \frac{1}{2\pi} \oint_{\partial R_\ell} d\boldsymbol{k} \cdot \boldsymbol{\nabla} \,\mathrm{Im}\ln\left(\frac{u_n^\alpha(\boldsymbol{k})}{u_n^\beta(\boldsymbol{k})}\right),$$

where $N_{n\ell}$ is the charge of a vortex at $\boldsymbol{k}_\ell$ [a zero point of $u_n^\alpha(\boldsymbol{k})$ in the Brillouin zone], R_ℓ is a region around $\boldsymbol{k}_\ell$ which does not include other zero points of the αth nor βth components, and ∂R_ℓ is the boundary. The arbitrariness in the choice of α and β corresponds to a freedom in gauge fixing. In other words, the gauge choice does not affect observables such as the Hall conductance. However, we need to fix the gauge to obtain physical quantities. Further, gauge dependent objects, e.g., the configuration of vortices, are helpful to understand the physics. As discussed below, the *sum rule* of the Hall conductance can be clearly understood by tracing the vortices. Here we comment on implication of the Chern number in the infinite size limit. If all states in the nth band are localized, the corresponding Chern number vanishes, $C_n = 0$. Therefore nonzero C_n means the existence of delocalized states. On the other hand, even if $C_n = 0$, we cannot exclude the existence of delocalized states in a rigorous sense. However, it is a situation in the Hall insulator, and we believe that all the states in the band become localized as the Anderson insulator.

There are several previous studies on the Hall conductance (Chern number) in this model [9,14]. In this paper, the plateau transition in a sufficiently weak magnetic field regime is focused on by the topological arguments and the numerical study. It has much connection with the continuum model. In order to explore the regime, the choice of the gauge is important. There are several choices for a given geometry of the system. Employing the *string* (anyon) gauge, we can study the topological nature of the Bloch wave functions in a sufficiently weak magnetic field regime [16]. An example for θ_{ij}'s in the string gauge is shown for a 3×3 square lattice in Fig. 1. The extension to the other geometries is straightforward. Choosing a plaquette S as a starting one, we draw outgoing arrows (strings) from the plaquette S. The θ_{ij} on a link ij is given by $2\pi\phi n_{ij}$, where n_{ij} is the number of strings which cut the link ij (the orientation is taken into account). Then it is clear that the magnetic flux is uniform except at the plaquette S. At the plaquette S, the condition of uniformity gives $e^{-i2\pi\phi(L_xL_y-1)} = e^{i2\pi\phi}$. It restricts the possible magnetic flux as

$$\phi = \frac{n}{L_xL_y} , \qquad n = 1, 2, \ldots, L_xL_y .$$

In the case of the standard Landau gauge in the x direction, the smallest compatible magnetic flux ϕ with the periodicity is $\phi = 1/L_x$. Then a system with a rectangular geometry is needed for a weak magnetic field regime. On the other hand, in our string gauge, it is $\phi = 1/L_xL_y$. For a square $L \times L$ geometry, it allows us to make use of L times smaller magnetic flux in the string gauge than the Landau gauge. The string gauge enables us to study a sufficiently weak magnetic regime.

We performed numerical diagonalizations of the above Hamiltonian and obtained the spectrum and the Bloch wave functions. Here the string gauge is employed. Zero points (vortices) of the Bloch wave functions and the winding number (charge) of each vortex are also calculated for all the energy bands. In Figs. 2, the configuration of vortices and their charges are shown with different randomness strengths W's. As discussed above, when the Fermi energy lies in an energy gap, the Hall conductance is quantized to an integer. The integers are shown in some energy gaps in Figs. 2. They are given by the sum of all the Chern numbers below each gap. Note that, although the energy gap has to close to change the Hall conductance as discussed below, the exact gap-closing points cannot be seen due to the lack of numerical accuracy. However, tracing the vortices, we can identify the gap closing points and some of them are shown by the triangles in Figs. 2.

Let us first summarize some features of the numerical results in Figs. 2. As the strength of randomness W is changed, vortices in each energy band move continuously, and the motion of a vortex forms a *vortex line*. However, with a small change of W, the Chern number of each energy band is stable. This is because the Chern number is the *topological invariant* of the energy band and the topology change is necessary to change it. As seen in Figs. 2, when the Chern number changes, the two energy bands touch. Then the two bands, in generic, touch at one point in the Brillouin zone and it forms a singularity. Near the point, the low-energy physics is described generally by massless Dirac fermions [17–19]. Associated with the appearance of the gap closing point, a vortex line passes through it and the Chern number in each energy band changes by ± 1. Therefore the Chern numbers of

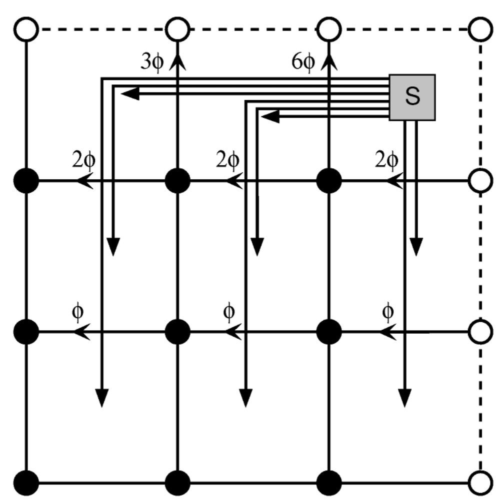

FIG. 1. The definition of the string (anyon) gauge for a 3×3 square system with periodic boundary condition.

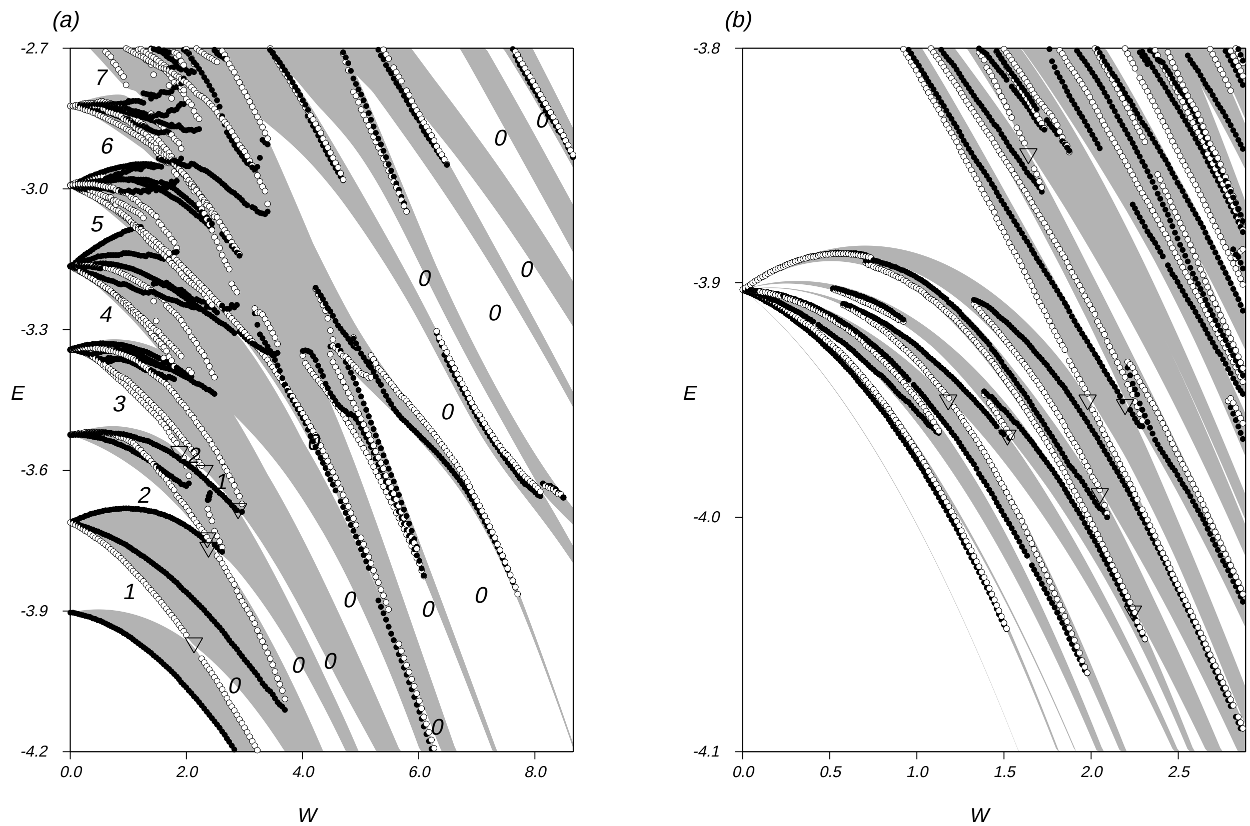

FIG. 2. Zero points (vortices) of the Bloch wave functions with their winding numbers (charges). The shaded regions show energy bands. Each circle denotes the position of a vortex as a function of randomness strength (W). The full circle means the charge $+1$ and the open circle -1. The Hall conductance is quantized to an integer, when the Fermi energy lies at an energy gap. The integers are shown at some gaps. The gap closing points can be seen within the numerical accuracy and some of them are shown by triangles. (a) The flux per plaquette is $\phi = \frac{1}{64}$ and the system size is $L_x \times L_y = 8 \times 8$. (b) The flux per plaquette is $\phi = \frac{1}{64}$ and the system size is $L_x \times L_y = 24 \times 24$. Only the region near the lowest Landau level is shown. The triangles correspond to the trajectory of the delocalized states. One can see the floating of the delocalized states relatively within the lowest Landau band.

the two energy bands do not change in total. This is the *sum rule* in our model [17–20]. It also leads to the *selection rule* $\Delta\sigma_{xy} = \pm 1$. As shown in Figs. 2, the overlap of the energy bands can also happen. Then the energy gap seems to be closed. However, there is still an energy gap in the Brillouin zone (the situation is similar to semimetals) and the Chern number of each energy band is well-defined. Therefore the vortex motion is still governed by massless Dirac fermions.

As discussed above, the basic observation of Figs. 2 is that the change of the Chern number in each energy band is described by massless Dirac fermions and, in general, obeys the selection rule $\Delta\sigma_{xy} = \pm 1$. In fact, one can see the transition $\sigma_{xy} = 3 \to 2 \to 1 \to 0$ in Fig. 2(a) where the electron density is fixed. However, the change of the *observed* σ_{xy} can break the rule (anomalous plateau transition). The definition of the *observed* σ_{xy} has subtle aspects. The σ_{xy} depends strongly on the randomness realization, the boundary condition, and the geometry in this transition region. Therefore a naive infinite size limit is not well-defined mathematically. One needs to define the *observed* Hall conductance by its average over different realizations. Then one can take the infinite size limit. Physically, in a realistic situation, there are several possibilities for the justification of the ensemble average. For example, (i) due to the finite coherence length, the system effectively decouples into several domains with different realizations of randomness, or (ii) since the thermal fluctuation exists, the σ_{xy} is averaged near the Fermi energy and the energy average may be replaced by the ensemble average [21]. In our model, small energy gaps can appear due to the existence of randomness. It is clearly seen in Fig. 2(b). The Hall conductance is quantized even when the Fermi energy lies in the small gap. However, these small gaps strongly depend on randomness realization and, after the ensemble average, the corresponding Hall conductance is generally not quantized. This is in contrast to the case when the Fermi energy lies in the Landau gap and the Hall conductance is quantized even after the ensemble average. In fact, after the ensemble average, the plateau transitions $\sigma_{xy} = 3 \to 2 \to 1 \to 0$ in Fig. 2(a) become $\sigma_{xy} = 3 \to 0$ $(\Delta\sigma_{xy} = 3)$. It demonstrates the transition $\Delta\sigma_{xy} \neq \pm 1$ (see also Fig. 3). Although the plateau transitions generally obey the selection

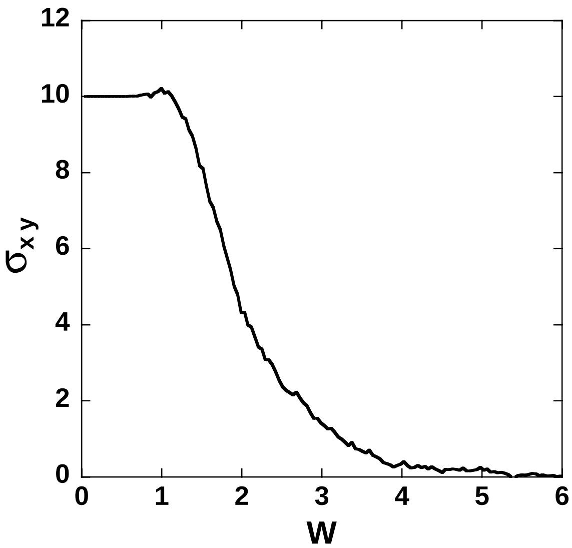

FIG. 3. The *observed* Hall conductance as a function of randomness strength where the electron density is fixed. The flux per plaquette is $\phi = 1/64$ and the system size is $L_x \times L_y = 8 \times 8$. The ensemble average is performed over 100 different realizations of randomness.

rule ($\Delta\sigma_{xy} = \pm 1$) for a given realization of randomness, the transition $\Delta\sigma_{xy} \neq \pm 1$ is observed due to the ensemble average. This is the anomalous transition.

Finally, we comment on the trajectory of delocalized states in the lowest Landau band. As seen in Fig. 2(b), the lowest Landau band splits into some subbands due to the existence of randomness. In our case ($\phi = 1/64$), before the collapse of the lowest Landau gap, the Hall conductance $\sigma_{xy} = 1$ when the Fermi energy lies in the gap. It implies that there are delocalized states in the lowest Landau band. In Fig. 2(b), when W is sufficiently small, only one of the subbands in the lowest Landau band carries a nonzero Chern number ($=+1$). Therefore we can assign the position of the delocalized states to the subband. In this way, the delocalized states can be traced. Through the energy gap closing, the Chern number changes and the delocalized states move from one subband to another. Further, when the gap closes, the position of the delocalized states can be identified with the gap closing points, i.e., the massless Dirac fermions. These gap closing points are shown by the triangles. It can be seen in Fig. 2(b) that, when the randomness strength is sufficiently small, the delocalized states float up relatively within the lowest Landau band. At the same time, the lowest Landau band broadens and goes downward in energy. However, before the delocalized states float across the lowest Landau gap, the gap collapses and the states disappear in pair with the other delocalized states falling down from the higher energy region. The pair annihilation point depends strongly on the randomness realization, and the energy is not observed after the ensemble average (it corresponds to the experimental situation). It has the same origin as the anomalous transition discussed above. Therefore we do not observe any sign of floating across the Landau gap.

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[21] On the other hand, when the temperature is sufficiently low that (i) the coherence length grows and the σ_{xy} for a given realization of randomness is observed or (ii) the thermal fluctuation is below the level spacing of the system, the σ_{xy} generally depends strongly on the randomness realization and the nonuniversal region should appear in the plateau transition with the change of electron density. Although this situation is not realistic in the macroscopic system, this nonuniversal region may appear in the mesoscopic system.