Decay Laws for Three-Dimensional Magnetohydrodynamic Turbulence

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Novel decay laws in three-dimensional incompressible magnetohydrodynamic turbulence are obtained by high-resolution numerical simulations with up to 512^3 modes and explained by a simple theoretical model. For the typical case of finite magnetic helicity H the energy decay is governed by the conservation of H and the decay of the energy ratio $\Gamma = E^V/E^M$. One finds the relation $(E^{5/2}/\epsilon H)\Gamma^{1/2}/(1 + \Gamma)^{3/2} = \text{const}$, where $\epsilon = -dE/dt$. Use of the numerical result that $\Gamma(t) \propto E(t)$ gives the asymptotic law $E \sim t^{-0.5}$ in good agreement with the numerical observations. For the special case $H = 0$ the energy decreases more rapidly $E \sim t^{-1}$.

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Many plasmas, especially in astrophysics, are characterized by turbulent magnetic fields, where the best-known example is the turbulence in the solar wind (actually the only one accessible to *in situ* observations, since there are practically no laboratory experiments). The decay of magnetic turbulence is an important theoretical issue, determining, for instance, the lifetime of star-forming clouds [1]. It is also connected with the question of fast magnetic reconnection. The convenient framework to describe such turbulence is magnetohydrodynamics (MHD). Here one ignores the actual complicated dissipation processes, which occur on the smallest scales and would usually require a kinetic treatment, assuming that the main turbulent scales are essentially independent thereof.

MHD turbulence has become a paradigm in turbulence theory, which has been receiving considerable attention. It is well known that 2D and 3D MHD turbulence have many features in common concerning, in particular, the cascade properties. In both cases there are three quadratic ideal invariants: the energy $E = \frac{1}{2}\int(v^2 + B^2)\,dV$, the cross helicity $K = \int \mathbf{v} \cdot \mathbf{B}\,dV$, and a purely magnetic quantity, the magnetic helicity $H = \int \mathbf{A} \cdot \mathbf{B}\,dV$ in 3D and the mean-square magnetic potential $H^\psi = \int \psi^2\,dV$ in 2D, which both exhibit an inverse cascade. Indeed many theoretical predictions do not distinguish between 2D and 3D, concerning, e.g., the tendency toward velocity and magnetic field alignment or the spectral properties. Thus it is not surprising that numerical studies of MHD turbulence have mostly been concentrated on two-dimensional simulations, where high Reynolds numbers can be reached much more readily (see, e.g., [2–7]). [In homogeneous turbulence the Reynolds number $\mathrm{Rm} = vL/\eta$ is defined using dynamic quantities of the turbulence $v = (E^V)^{1/2}$ and $L = E^{3/2}/\epsilon$, where ϵ is the energy dissipation rate. In this Letter, the precise definition is not important; higher Reynolds number simply means lower η, initial conditions being similar.] While 2D simulations are now being performed with up to $N^2 = 4096^2$ modes (or, more accurately, collocation points) [7], studies of 3D MHD turbulence have until now been restricted to relatively low Reynolds numbers using typically $N^3 = 64^3$ modes, e.g., [8,9], which precludes an inertial range scaling behavior. Also, in Ref. [10], where a somewhat higher Reynolds number could be reached by using 180^3 modes, attention was focused primarily on the process of turbulence generation from smooth initial conditions and the properties of the prominent spatial structures, current, and vorticity sheets.

In this Letter we present results of a numerical study of freely decaying 3D MHD turbulence with spatial resolution up to 512^3 modes. We discuss the decay laws of the integral quantities, in particular, the energy E and the ratio of kinetic and magnetic energies $\Gamma = E^V/E^M$, and their dependence on the quasiconstant value of H. The energy decay is found to follow a simple law, which is determined by $\Gamma(t)$ and H. While most previous studies have been restricted to the case of negligible magnetic helicity $H \simeq 0$, we focus attention on the properties of the turbulence for finite H, which is more typical for naturally occurring magnetic turbulence. We find that for finite H the energy decays significantly more slowly than for $H \simeq 0$. This behavior is also due to the rapid decrease of the energy ratio Γ, which has the same decay time as the energy itself.

The 3D incompressible MHD equations, written in the usual units,

$$\partial_t \mathbf{B} - \nabla \times (\mathbf{v} \times \mathbf{B}) = \eta_\nu (-1)^{\nu-1} \nabla^{2\nu} \mathbf{B}, \qquad (1)$$

$$\partial_t \mathbf{w} - \nabla \times (\mathbf{v} \times \mathbf{w}) - \nabla \times (\mathbf{j} \times \mathbf{B}) = \mu_\nu (-1)^{\nu-1} \nabla^{2\nu} \mathbf{w}, \qquad (2)$$

$$\mathbf{w} = \nabla \times \mathbf{v}, \qquad \mathbf{j} = \nabla \times \mathbf{B},$$

are solved in a cubic box of size 2π with periodic boundary conditions. The numerical method is a pseudospectral scheme with spherical mode truncation as conveniently used in 3D turbulence simulations instead of full dealiasing by the 2/3 rule chosen in most 2D simulations. The effect of aliasing errors has been discussed by Orszag [11] (see also [12]). Initial conditions are

$$\mathbf{B}_\mathbf{k} = \mathbf{a}e^{-k^2/k_0^2 - i\alpha_\mathbf{k}}, \qquad \mathbf{v}_\mathbf{k} = \mathbf{b}e^{-k^2/k_0^2 - i\beta_\mathbf{k}}, \tag{3}$$

which are characterized by random phases $\alpha_\mathbf{k}$, $\beta_\mathbf{k}$ and satisfy the conditions $\mathbf{k} \cdot \mathbf{B}_\mathbf{k} = \mathbf{k} \cdot \mathbf{v}_\mathbf{k} = 0$ as well as $E = 1$ and $\Gamma = 1$. Further restrictions on $\mathbf{B}_\mathbf{k}$ and $\mathbf{v}_\mathbf{k}$ arise by requiring specific values of H and K, respectively. The wave number k_0, the location of the maximum of the initial energy spectrum, is chosen as $k_0 = 4$, which allows the inverse cascade of $H_\mathbf{k}$ to develop freely during the simulation time of 10–20 eddy turnover times. This implies a certain loss of inertial range, i.e., a reduction in Reynolds number, but the sacrifice is unavoidable in the presence of inverse cascade dynamics. Choosing $k_0 \sim 1$ would lead to magnetic condensation in the lowest-k state, which would affect the entire turbulence dynamics. We have used both normal diffusion $\nu = 1$ and hyperdiffusion $\nu = 2$. Apart from the fact that inertial ranges are wider and H is still better conserved for $\nu = 2$ than for $\nu = 1$, no essential differences are found between the two cases. The generalized magnetic Prandtl number η_ν/μ_ν has been set equal to unity. Table I lists the most important parameters of the simulation runs.

The energy decay law is a characteristic property of a turbulent system. In hydrodynamic turbulence the decay rate depends on the energy spectrum at small k. Assuming time invariance of the Loitsianskii integral $\mathcal{L} = \int_0^\infty dl\, l^4 \langle v_l(x + l) v_l(x) \rangle$, the energy has been predicted to follow the similarity law $E \sim t^{-10/7}$ [13]. The invariance of $\mathcal{L}$ has, however, been questioned (see, e.g., [14]). Both closure theory [15] and low Reynolds number simulations [9] yield a significantly slower decrease, $E \sim t^{-1}$. Experimental measurements of the energy decay law t^{-n} are rather difficult and do not give a uniform picture, n ranging between 1.3 [16] and 2 [17].

The invariance of the Loitsianskii integral has recently also been postulated for MHD turbulence [5], where $\mathcal{L}_{\mathrm{MHD}}$ is defined in analogy to $\mathcal{L}$ in terms of the longitudinal correlation function $\langle z_l^\pm(x + l) z_l^\pm(x) \rangle$ of the Elsaesser fields $\mathbf{z}^\pm = \mathbf{v} \pm \mathbf{B}$. Since $z^2 \sim E$, this assumption gives $\mathcal{L}_{\mathrm{MHD}} \sim L^5 E = \text{const}$, where L is the integral scale length of the turbulence. In addition, the expression for the energy transfer $dE/dt = -\epsilon \sim -z^4/LB_0$ was used, which formally accounts for the Alfvén effect [18,19]. These relations give $(dE/dt)B_0/E^{11/5} = \text{const}$ and, hence, $E \sim t^{-5/6}$, treating B_0 as constant. One may, however, argue that the Alfvén effect is only important on small scales $l \ll L$, while on the scale L of the energy-containing eddies B_0 is not constant but $B_0 \sim E^{1/2}$ (except for the case that B_0 is an external field, which would, however, make the turbulence strongly anisotropic); hence, $\epsilon \sim E^{3/2}/L$, which would give the same result $n = 10/7$ as predicted for hydrodynamic turbulence. Low-resolution numerical simulations [9] indicate $n \simeq 1$, which is also found in recent simulations of compressible MHD turbulence [1].

TABLE I. Summary of the simulation runs. The value of $H = 0.28$ corresponds to the maximum value for the given spectrum (3), $H \leq H_{\max} \simeq E/k_0$. The initial alignment is measured by the correlation $\rho_0 = K/E$. The Reynolds numbers are taken at $t = 4$. No numbers are given for hyperresistivity $\nu = 2$, where the classical Reynolds number is not defined.

Run No.	N	ν	η_ν	Rm	H	ρ_0	$t_{\max}$
1	256	1	10^{-3}	1600	0.19	0.04	18.5
2	512	1	3×10^{-4}	6400	0.19	0.04	10
3	256	2	10^{-6}	$\cdots$	0	0.05	20
4	256	2	10^{-6}	$\cdots$	0.11	0.05	10
5	256	2	10^{-6}	$\cdots$	0.19	0.04	20
6	512	2	3×10^{-8}	$\cdots$	0.19	0.04	10
7	256	2	10^{-6}	$\cdots$	0.25	0.04	10
8	256	2	10^{-6}	$\cdots$	0.28	0.03	10
9	256	2	10^{-6}	$\cdots$	0.19	0.38	10
10	256	2	10^{-6}	$\cdots$	0.19	0.71	10

For finite magnetic helicity H provides a constant during energy decay, which for high Reynolds number is more robust than the questionable invariance of the Loitsianskii integral. It is true that in contrast to the 2D case, where E^M and H^ψ are tightly coupled, such that $E^M \neq 0$ implies $H^\psi \neq 0$, in 3D a state with $H = 0$ and finite magnetic energy is possible. But this is only a special and nontypical case, since in nature magnetic turbulence usually occurs in rotating systems, which give rise to finite magnetic helicity.

If the process of turbulence decay is self-similar, which also implies that the energy ratio Γ remains constant, the energy decay law follows from a simple argument [20]. With the integral scale length $L = E^{3/2}/\epsilon$, the dominant scale of the energy-containing eddies, we have

$$H \simeq E^M L \sim EL, \tag{4}$$

since owing to the assumed self-similarity $E^M \sim E^V \sim E$. Inserting L gives

$$-\frac{dE}{dt} = \epsilon \sim \frac{E^{5/2}}{H}, \tag{5}$$

which has the similarity solution $E \sim t^{-2/3}$ (this behavior has also been predicted by Hatori [21], though using a different approach). In Fig. 1 the ratio $E^{5/2}/(\epsilon H)$ is plotted for the runs from Table I with $H \neq 0$ and small initial correlation ρ_0. The figure shows that this quantity is not constant, but increases in time. Moreover, there is a significant spread of the different curves. Integration yields a slower asymptotic energy decay, $n \simeq 0.5$–0.55. [The log-log representation of $E(t)$, often given in the literature to make a power law behavior visible, is misleading, since the major part of such a curve refers to the transition period of turbulence generation. The solution $(t - t_*)^{-n}$ approaches the power law t^{-n} only asymptotically for $t \gg t_*$, where t_* is not accurately

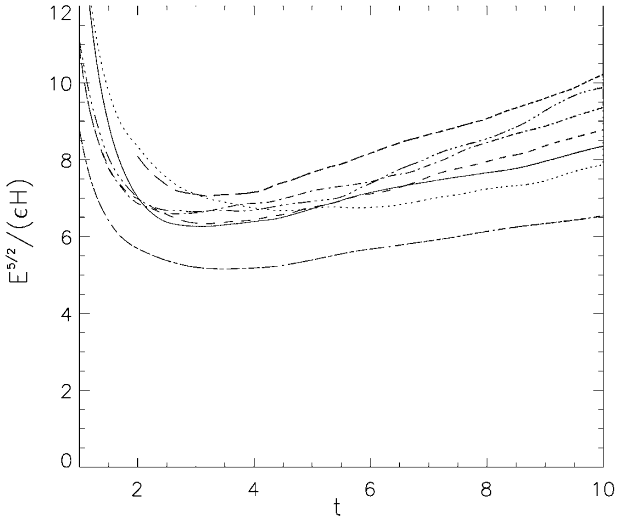

FIG. 1. Energy decay law, displayed in the differential form $E^{5/2}/\epsilon H$ for the runs 1, 2, 4, 5, 6, 7, and 8 in Table I. The increase in time indicates an energy decrease slower than $t^{-2/3}$, typically $t^{-0.5}$.

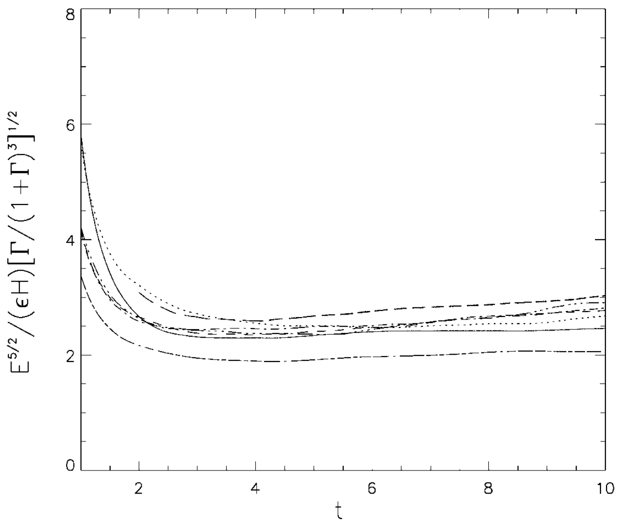

FIG. 2. Energy decay law in differential form $(E^{5/2}/\epsilon H)\Gamma^{1/2}/(1 + \Gamma)^{3/2}$ for the same runs as in Fig. 1. The lowest curve, which falls somewhat outside the main curve bundle, corresponds to the run with the smallest Reynolds number (run 1), where conservation of H is least good.

known. We therefore prefer to plot the decay law in the primary differential form.]

We can attribute this discrepancy to the fact that the turbulence does not decay in a fully self-similar way. Indeed the energy ratio Γ is found to decrease rapidly, in contrast to the 2D case, where Γ is quasiconstant decaying at most logarithmically [3,4]. (The ratio of viscous and resistive dissipation $\epsilon^\mu/\epsilon^\eta$, however, remains constant just as in the 2D case [4], which simply reflects the basic property that dissipation takes place in current sheets and that these are also vorticity sheets, i.e., the location of viscous dissipation.) Let us incorporate the dynamic change of Γ in the theory of the energy decay. Assuming that the most important nonlinearities arise from the $\mathbf{v} \cdot \nabla$ contributions in the MHD equations, Eq. (5) is replaced by

$$\epsilon \sim (E^V)^{1/2}\,\frac{E}{L} = \frac{\Gamma^{1/2}}{(1+\Gamma)^{3/2}}\,\frac{E^{5/2}}{H}\,, \tag{6}$$

using the relation (4). Figure 2 shows that $(E^{5/2}/\epsilon H)\Gamma^{1/2}/(1 + \Gamma)^{3/2}$ is indeed nearly constant for $t > 2$, when turbulence is fully developed, and the spread of the different curves in Fig. 1 is strongly reduced. Hence, relation (6) is generally valid for finite magnetic helicity. Figure 2 also shows that the turbulence decay is practically independent of the magnitude of the dissipation coefficients, i.e., the Reynolds number, as well as of the character of the dissipation ($\nu = 1$ or 2).

Also the time evolution of the energy ratio Γ exhibits a uniform behavior which is demonstrated in Fig. 3. Since the initial nonturbulent relaxation of Γ depends on H, we have normalized $\Gamma(t)$ to the value at $t = 1$, where the dissipation rate assumes its maximum and the turbulent decay starts. Moreover, we find that $\Gamma(t)$ is proportional to $E(t)$, $\Gamma \simeq cE/H$, $c = 0.1$–0.15, as seen in Fig. 4, where $\Gamma/(E/H)$ is plotted. Inserting this result in Eq. (6), we obtain the differential equation for E, which in the asymptotic limit $\Gamma \ll 1$ becomes

$$-\frac{dE}{dt} \simeq 0.5\,\frac{E^3}{H^{3/2}}\,, \tag{7}$$

with the similarity solution $E \sim t^{-0.5}$. For finite Γ the theory predicts a somewhat steeper decay flattening asymptotically to $t^{-0.5}$ as Γ becomes small, which is exactly the behavior of $E(t)$ observed in the simulations. The relation $\Gamma \propto E$ also gives the similarity law for the kinetic energy $E^V \sim t^{-1}$.

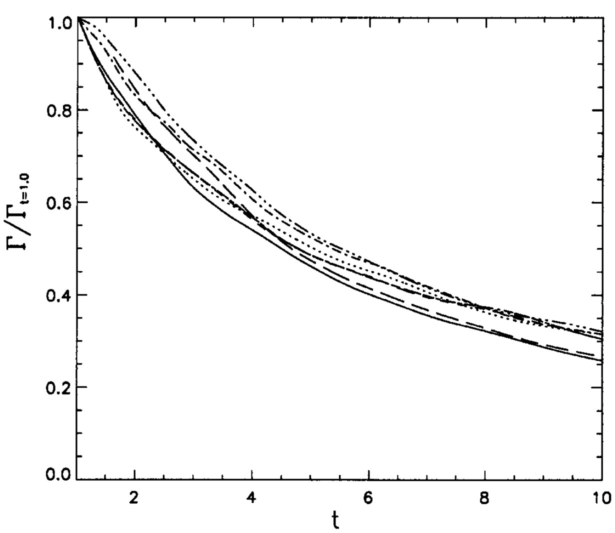

FIG. 3. Energy ratio $\Gamma(t)$ normalized to the values at $t = 1$ for the same runs as in Fig. 1.

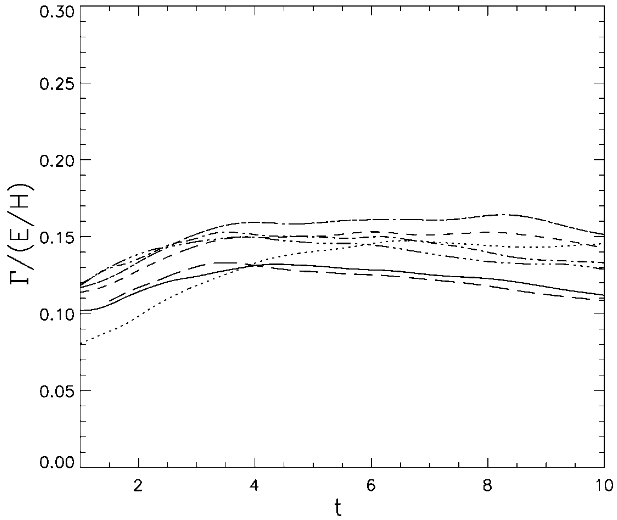

FIG. 4. $\Gamma/(E/H)$ for the same runs as in Fig. 1 demonstrating the proportionality $\Gamma \propto E$.

The rapid decay of the energy ratio Γ implies that the system decays toward a static magnetic state, which is consistent with Taylor's conjecture [22]. The $t^{-0.5}$ decay of the magnetic energy continues, as long as the turbulent dynamics is still active. In the asymptotic static state the energy decays, of course, much more slowly.

This theory does not apply to the special case $H = 0$. Here we find indeed a different decay law, $E \sim t^{-1}$ from run 3, which is consistent with previous simulations at lower Reynolds numbers [9]. The transition to the slower decay for finite H occurs at relatively small values, 10%–20% of the maximum possible value.

We have also studied the effect of an initial velocity–magnetic-field alignment $\rho_0 = K/E$. For small $\rho_0 < 0.1$, the alignment, after increasing initially, tends to saturate at some small value, which is due to the fact that K is less well conserved than H. For higher $\rho_0 > 0.3$ (runs 9 and 10 in Table I) the alignment becomes very strong, which as expected slows down the energy decay drastically.

In conclusion, we have presented a new phenomenology of the energy decay in 3D incompressible MHD turbulence, which agrees very well with direct numerical simulations at relatively high Reynolds numbers. We consider, in particular, the case of finite magnetic helicity H, which is typical for naturally occurring magnetic turbulence. The energy decay is governed by the conservation of H and the time evolution of the energy ratio $\Gamma = E^V/E^M$. We find that the relation $(E^{5/2}/\epsilon H)\Gamma^{1/2}/(1 + \Gamma)^{3/2} = \text{const}$ is satisfied for most H values and is independent of the magnitude of the dissipation coefficients and the order of the diffusion operator, provided the Reynolds number is sufficiently high such that H is well conserved. The kinetic energy is found to decrease more rapidly than the magnetic one in contrast to the behavior in 2D, in particular, we find $\Gamma \propto E$. This proportionality leads to a simple energy decay law, $-dE/dt \sim E^3$, or $E \sim t^{-0.5}$. We also obtain the similarity law for the kinetic energy $E^V \sim t^{-1}$. For the special case $H = 0$ the energy decays more rapidly, $E \sim t^{-1}$, which agrees with previous simulations at lower Reynolds numbers. The transition to the finite-H behavior occurs at relatively small values of H.

Results concerning the spatial scaling properties of 3D MHD turbulence will be published in a subsequent paper.

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