Measure Synchronization in Coupled Hamiltonian Systems

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(Received 28 May 1999)

The dynamics of two coupled Hamiltonian systems is shown to exhibit a transition to coherent evolution as the strength of coupling is increased. Above this transition the two systems are not strictly synchronized, but their orbits cover the same region of the (individual) phase space with identical invariant measures. This *measure synchronization* is numerically observed in maps and time-continuous systems, for both regular and chaotic evolution, and can be analytically derived for regular systems in the action-angle representation, which suggest that it is a generic form of coherent evolution in Hamiltonian dynamics. Analytical results show that the transition involves logarithmic singularities.

PACS numbers: 05.45.Xt, 05.45.Pq, 05.20.–y, 05.50.+q

Because of their role in the analysis of complex behavior in natural phenomena, coupled dynamical systems have attracted great attention in the last decade. Arrays of coupled elements have been used as models of complex extended systems [1], and globally coupled ensembles [2,3] have been used to represent systems driven by long-range interactions, such as neural networks [4]. It has been shown that in globally coupled dissipative systems formed by identical elements the typical form of collective behavior is synchronization [2–5]. Above a critical value of the coupling strength, all the elements approach asymptotically the same orbit. The appearence of this kind of coherent, condensed state is analogous to a phase transition, and the critical point is determined by the Lyapunov exponents of the individual dynamics of each element [3,5].

As described above, synchronization is not possible in conservative systems. In Hamiltonian ensembles, Liouville's theorem prevents the full collapse of the orbits, since volume must be preserved in phase space. Coupled Hamiltonian elements, therefore, should display a different class of collective evolution [6]. Though globally coupled Hamiltonian ensembles could provide a suitable model for studying many-body mechanical systems with long-range interactions, the analysis of their collective behavior has received relatively less attention. However, it has recently been shown that they can exhibit forms of weak synchronization [6–8].

The goal of this Letter is to point out a generic form of collective behavior in coupled Hamiltonian systems, which we have numerically found to occur both in time-discrete and time-continuous dynamics, for ensembles of two or more elements with regular or chaotic orbits. We have also been able to analytically find such behavior for a two-element system in the action-angle representation, i.e., for nonchaotic orbits.

In order to describe this form of collective behavior by means of an example, let us first consider a coupled pair of identical two-dimensional (2D) Hamiltonian elements with coordinates (x_1, y_1) and (x_2, y_2), respectively. The corresponding phase space is thus four dimensional. However, since we are interested in analyzing the effects of coupling on the individual dynamics of each element, it is convenient to "project" the state of the full system onto the one-element space. More precisely, we study the two trajectories $(x_i(t), y_i(t))$ $(i = 1, 2)$ on the (x, y) plane. We find that, for initial conditions on different trajectories, the regions of the plane covered by the two orbits do not intersect when the coupling intensity is small. As the coupling intensity grows, however, these regions become broader and approach each other. Then, at a critical coupling the two regions suddenly merge into a single region of the plane. From then on, the two orbits will (almost) always lie in the same domain. Although they never coincide simultaneously at the same point of the plane, each of them eventually passes arbitrarily close to any point visited by the other. A closer analysis of the orbits above the critical coupling shows that they define identical invariant measures [9] on the portion of the plane that they share. Thus, we call this form of coherent evolution *measure synchronization* (here *synchronization* is understood in a broad sense). Measure synchronization is observed for any coupling larger than the critical point, except at some narrow windows where it breaks down and the orbits separate from each other.

To avoid discretization errors, most of our numerical realizations of coupled Hamiltonian systems involve maps. Hamiltonian maps are characterized by their simplectic structure [10]. A well-known example is the 2D standard map on the torus $(0, 2\pi] \times (0, 2\pi]$,

$$\begin{aligned} x(t+1) &= x(t) + y(t) + \alpha \sin x(t), \qquad \mod 2\pi\,, \\ y(t+1) &= y(t) + \alpha \sin x(t), \qquad \mod 2\pi\,, \end{aligned} \tag{1}$$

where α is a nonlinearity parameter. For moderately large values of α this map exhibits both regular quasiperiodic orbits and chaotic evolution, depending on the initial conditions [10].

The form of global coupling usually employed with maps [3], which is inherently dissipative, cannot be applied to Hamiltonian systems without destroying their conservative character. Instead, we introduce coupling by means of an additional Hamiltonian map, to be successively applied to the system at each time step, after the individual evolution map (1) has acted. In other words, we apply at each time step the composition of the individual evolution and coupling. If both maps are Hamiltonian, their composition is in turn Hamiltonian. Global coupling between N maps (x_i, y_i) $(i = 1, \ldots, N)$ is here given by the simplectic recursion equations

$$\begin{aligned} x_i(t+1) &= x_i(t), \\ y_i(t+1) &= y_i(t) \\ &\quad + \frac{K}{N}\sum_j \sin[x_j(t) - x_i(t)], \quad \mathrm{mod}\ 2\pi, \end{aligned} \tag{2}$$

where K is the coupling intensity. This form of coupling is similar to the interaction usually imposed on systems of globally coupled oscillators in the phase approximation [2,7] and on the global coupling used in continuous Hamiltonian systems [6], where it represents an effective interaction potential.

We have begun by considering a pair of coupled standard maps with $\alpha = -1.2$. Recalling that in Hamiltonian dynamical systems initial conditions act as independent additional parameters for each orbit, we have first taken the initial conditions of the two maps in the zone of regular, nonchaotic evolution, $(x_1(0), y_1(0)) = (0.2807, -0.0802)$ and $(x_2(0), y_2(0)) = (-0.1471, 0.0134)$. In the absence of coupling, $K = 0$, these initial conditions correspond to two different quasiperiodic orbits, which cover closed curves in the (x, y) plane (see Fig. 1a). For $K > 0$ these two curves become 2D rings, as a result of the increase of the effective dimension of the whole Hamiltonian system. As K grows both rings widen in such a way that the external border of the inner ring approaches the internal border of the outer ring. Note in Fig. 1b that the respective densities are larger at those borders. At the point $K_c \approx 3.18 \times 10^{-3}$ at which the rings would come into contact a sudden change occurs. As advanced above, the two regions merge into a single ring which is now covered by both orbits with the same density, i.e., defining the same invariant measure (Fig. 1c). Just above K_c, the 2D volume of this single ring is given by the sum of the volumes of the two rings just below the transition. Thus, it can be said that at the transition each element invades the domain occupied by the other. Measure synchronization is maintained for larger coupling intensities, except for some narrow intervals of K where the regions occupied by the two elements suddenly separate from each other (Fig. 1d). These *desynchronization windows*, which are probably related to occasional frequency resonances, are reminiscent of the stability windows of chaotic systems. We have observed the same kind of qualitative behavior for many other pairs of regular orbits, including the islands of stable evolution. The critical coupling intensity K_c, however, depends strongly on the chosen initial conditions.

Coupling a chaotic orbit with a regular orbit or two chaotic orbits produces essentially the same scenario. We have taken initial conditions from the chaotic "sea" surrounding one of the main chain of stability islands, where the orbits cover a 2D ringlike region even for $K = 0$. As K grows, the regions covered by the orbits widen and eventually merge, as described for two regular orbits. It is interesting to stress that coupling has no effect on the regularity of the evolution, at least, in connection with the transition to measure synchronization. Measurements of the Lyapunov exponents show that, typically, the coupled system is chaotic for small K if at least one of the orbits is chaotic for $K = 0$ and is regular if the two uncoupled orbits are regular. For larger K, the evolution can alternate between chaos and regularity, but the transitions are not related to changes in synchronization. Thus, Lyapunov exponents seem to be insensitive to the synchronization state and are therefore not suitable order parameters for characterizing the transition.

Along both the regular and the chaotic orbits considered above, which evolve on ringlike regions of the (x, y) plane, it is possible to define a phase which approximately describes the state of each element at a given time. This can be done on the same lines as the phase approximation used for coupled limit-cycle systems [2]. Under an appropriate (linear) variable change, the linear part of the standard map (1) can be transformed into a pure rotation. Up to nonlinear corrections, thus, the rings covered by the orbits are circular. The phase $\phi_i(t)$ of each element can be simply defined as the angle determined by the segment from the origin to the position of the element at time t and, for instance, the y axis. Numerical realizations show that the difference $\phi(t) = \phi_1(t) - \phi_2(t)$ exhibits a

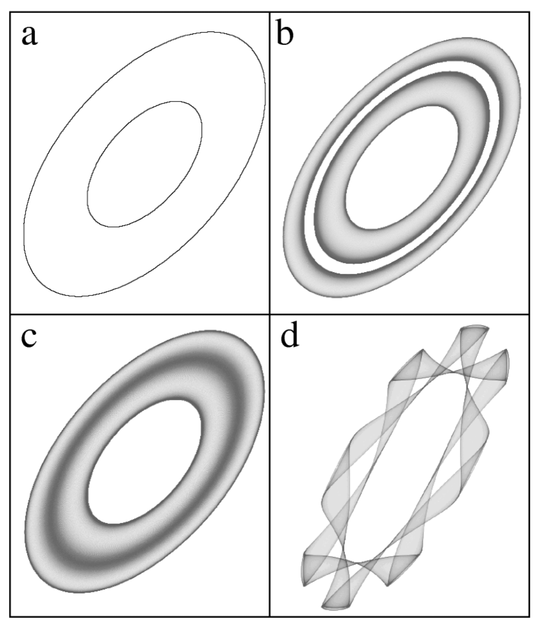

FIG. 1. Density plots of the invariant measures defined by two coupled standard maps on the one-element plane, for four values of the coupling intensity. (a) $K = 0$, no coupling; (b) $K = 3.1 \times 10^{-3}$, below the transition point; (c) $K = 3.4 \times 10^{-3}$, above the transition point; (d) $K = 1.2$, within a desynchronization window.

qualitative change when the transition to measure synchronization takes place. In fact, below the critical point, when the two orbits evolve in nonoverlaping domains, the phase difference grows (or decreases), in average, linearly with time, $\phi(t) \approx \eta t + \phi_0$. Naturally, this is a consequence of the fact that the orbits have different main frequencies. On the other hand, when the two orbits are synchronized and share the same domain, the phase difference oscillates with a characteristic frequency f around a constant value, $\phi(t) = \phi(t + 2\pi/f)$. This suggests defining two order parameters for the synchronization transition, namely,

$$\eta = |\lim_{t\to\infty} t^{-1}\phi(t)|, \tag{3}$$

and f as the main Fourier component of $\phi(t)$. Note that f vanishes below the transition whereas η vanishes above the transition. Figure 2 shows measurements of the two order parameters as a function of the coupling intensity for the pair of regular orbits considered above. Near $K = K_c$ both show a characteristic critical-like behavior, although the way they approach zero at the critical point is particularly abrupt. In fact, we were unable to determine a critical exponent from the numerical data. The analytical results presented below suggest that this transition could actually correspond to a nonalgebraic, logarithmic singularity.

Measure synchronization can be analytically explained under simplified conditions, for a pair of coupled two-dimensional time-continuous Hamiltonian systems in the action-angle (θ, ω) representation [10], i.e., for regular orbits. In the following we show that many of the features observed in numerical realizations are reproduced, at least qualitatively, by this approach. Consider the Hamiltonian $\mathcal{H}(\theta_1, \theta_2, \omega_1, \omega_2) = (\omega_1^2 + \omega_2^2)/2 - K\cos(\theta_2 - \theta_1)/2$. This corresponds to two identical dynamical elements with Hamiltonian $\mathcal{H}(\theta, \omega) = \omega^2/2$ coupled through their angle variables with an interaction weighted by the constant K. The corresponding canonical equations are

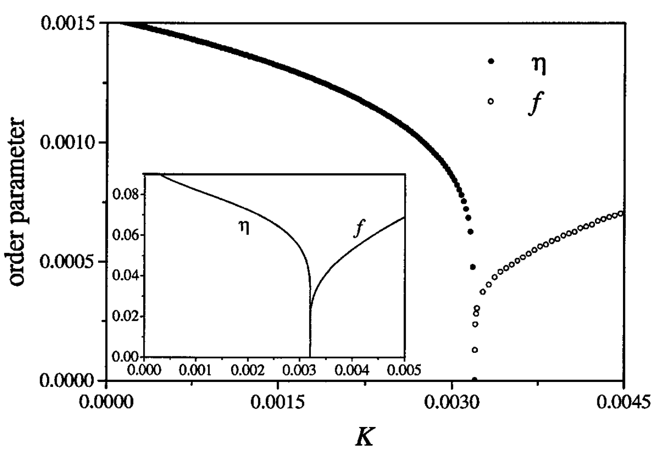

FIG. 2. The order parameters η and f defined below and above the critical point $K_c \approx 3.18 \times 10^{-3}$ as a function of the coupling intensity, from numerical realizations for two coupled standard maps. Inset: The order parameters for system (4), as calculated from the exact analytical solution for a set of initial conditions with the same critical point as above.

$$\begin{aligned} \dot\theta_1 &= \omega_1, \qquad \dot\omega_1 = \frac{K}{2}\sin(\theta_2 - \theta_1), \\ \dot\theta_2 &= \omega_2, \qquad \dot\omega_2 = \frac{K}{2}\sin(\theta_1 - \theta_2). \end{aligned} \tag{4}$$

For $K = 0$, θ_i grows at constant rate ω_i. In the 2D representation where θ is the polar angle and ω is the radius, the two uncoupled orbits are thus circular, with constant radius and angular velocity.

Taking into account that, from Eqs. (4), $\omega_1(t) + \omega_2(t) = \omega_1(0) + \omega_2(0) = \Omega$ (constant) and thus $\theta_1(t) + \theta_2(t) = \theta_1(0) + \theta_2(0) + \Omega t$, the equations can be reduced to a 2D system for the variables $\xi(t)$ and $\nu(t)$, defined by $\xi(t) = \theta_1(t) - \theta_2(t)$ and $\omega_{1,2}(t) = \omega_{1,2}(0) \pm \nu(t)$:

$$\dot\xi = \omega_0 + 2\nu, \qquad \dot\nu = -\frac{K}{2}\sin\xi, \tag{5}$$

with $\omega_0 = \omega_1^0 - \omega_2^0$. Differentiating the first of these equations and replacing the second, we get $\ddot\xi + K\sin\xi = 0$, i.e., Newton's equation for a pendulum, which can be exactly solved in terms of elliptic functions for given initial conditions $\xi(0) = \theta_1(0) - \theta_2(0)$ and $\dot\xi(0) = \omega_1(0) - \omega_2(0)$. A first integral of this equation is $E = \dot\xi^2/2 - K\cos\xi$, where the constant $E = \dot\xi(0)^2/2 - K\cos\xi(0)$ is the total energy. As is well known, a pendulum can exhibit two qualitatively different kinds of orbits. For large energies ($E > K$) the orbits are rotations, whereas for small energies ($E < K$) the pendulum oscillates around the equilibrium point $\xi = 0$. In the rotations, ξ increases (or decreases) monotonically while $|\dot\xi|$ oscillates between the two extreme values $\sqrt{2(E \pm K)}$. On the other hand, in the oscillations both ξ and $\dot\xi$ evolve symmetrically around zero. The orbit with $E = K$ acts as a separatrix between the two regimes. For given initial conditions for θ_i and ω_i this separatrix occurs at a critical value of the coupling intensity, given by

$$K_c = \frac{\dot\xi(0)^2}{2[1 + \cos\xi(0)]}. \tag{6}$$

Rotations and oscillations occur for $K < K_c$ and $K > K_c$, respectively. Note that $E = K_c + (K_c - K)\cos\xi(0)$.

Let us now translate these results in terms of the evolution of the original variables, in particular, of $\omega_1(t)$ and $\omega_2(t)$. These can be written as $\omega_{1,2} = (\Omega \pm \dot\xi)/2$. For sufficiently small values of K, $\dot\xi$ performs small oscillations with amplitude of order K/K_c around $\sqrt{E} \approx \sqrt{2K_c}$. The frequencies ω_1 and ω_2 oscillate then around two well-separated values $\Omega/2 \pm \sqrt{K_c/2}$. As long as $K < K_c$, in fact, the two frequencies vary within nonoverlaping intervals $[\Omega/2 + \sqrt{(E - K)/2}, \Omega/2 + \sqrt{(E + K)/2}]$ and $[\Omega/2 - \sqrt{(E - K)/2}, \Omega/2 - \sqrt{(E + K)/2}]$, respectively. When K tends to the critical value, however, both the lower limit of the upper interval and the upper limit of the lower interval approach $\Omega/2$. As soon as K overcomes K_c, a sudden change occurs. Now,

$\dot{\xi}$ oscillates around zero with amplitude $\sqrt{2(E+K)}$ and thus both ω_1 and ω_2 vary within the same interval $[\Omega/2 + \sqrt{(E+K)/2}, \Omega/2 - \sqrt{(E+K)/2}]$. The two intervals of frequency variation have completely merged into a single interval.

Recalling that ω_i represents the radius of the orbit of each element, we see that the continuous system (4) closely reproduces the behavior observed for two coupled standard maps. For small coupling intensity, the two orbits evolve on nonoverlaping circular rings whose borders approach each other as K grows. At the critical point the two rings come into contact and collapse into a single ring covered by both orbits. Note that (6) predicts that the critical coupling would strongly depend on the initial conditions, as observed in the numerical realizations for standard maps. It can be readily shown from the symmetry of Eqs. (4) that, once the two elements share the same region, their orbits define identical invariant measures on it. Moreover, the measure profiles predicted below and above the transition are qualitatively similar to the ones observed for the standard maps, Figs. 1b and 1c, with maxima in the inner zones of the accessible regions. Details will be given in a forthcoming publication [11].

For system (4), it is also possible to exactly obtain the order parameters defined above in terms of the phases along the orbits. In fact, the phase difference is here $\phi(t) \equiv \xi(t)$, which can be exactly given in terms of elliptic integrals. We obtain $\eta = \pi\sqrt{E+K}/\sqrt{2}\,\mathcal{K}[\sqrt{2K/(E+K)}]$ and $f = \pi\sqrt{E+K}/\sqrt{8}\,\mathcal{F}[\arccos(-E/K)/2, \sqrt{2K/(E+K)}]$, where $\mathcal{F}(\varphi,\kappa)$ is the elliptic integral of the first kind, and $\mathcal{K}(\kappa) = \mathcal{F}(\pi/2,\kappa)$ is the corresponding complete elliptic integral. Near the transition point, both order parameters exhibit the same critical behavior:

$$\eta \approx 2f \approx \pi\sqrt{K_c}/|\ln|K - K_c||\,. \tag{7}$$

This logarithmic singularity contrasts with the typical power-law dependence $|K - K_c|^\gamma$ near critical points in other transitions. Remarkably enough, it is the same critical behavior as found in the 2D Ising model [12]. The order parameters η and f vanish for $K \to K_c$ much more abruptly than in ordinary transitions, corresponding to a singular limit $\gamma \to 0^+$ for the critical exponent. This is again in agreement with the results for coupled standard maps, shown in Fig. 2. In the inset to that figure, we show η and f for system (4) near the critical point. Note the close similarity with the numerical results.

The fact that measure synchronization occurs in the action-angle system (4) strongly suggests that this kind of coherent evolution is a general property of globally coupled Hamiltonian systems, at least, for regular orbits in time-continuous dynamics. Indeed, any regular Hamiltonian system can be given in the action-angle representation by means of a suitable canonical transformation. Although the form of coupling in (4) is not generic, it is expected to represent more general situations. Critical properties depend in fact on the form of the interaction potential at its maxima only. Any regular potential with quadratic maxima should therefore give rise to the same kind of transition. We have numerically checked that the same critical phenomenon occurs for a pair of coupled point masses in the usual space-momentum variables, with Hamiltonian $\mathcal{H} = (p_1^2 + p_2^2)/2 + V(x_1) + V(x_2) + K(x_1 - x_2)^2/2$ and $V(x) = -x^2/2 + x^4/4$ [6]. Moreover, the numerical results quoted above show that chaotic orbits exhibit identical behavior.

Finally, we have performed preliminary numerical realizations of several ($N > 2$) standard maps coupled as in (2). Under certain conditions, we have observed that the elements show measure synchronization by pairs as the coupling intensity is increased. Typically, the closer two orbits are for $K = 0$ the smaller is the value of K at which they merge. This behavior is, however, not completely general. For given initial conditions there are pairs of orbits which simply overlap as coupling increases. Nevertheless, we have always found a value of K at which *all* the elements share a single region of the (x, y) plane. For sufficiently strong coupling, then, measure synchronization does occur and involves the whole Hamiltonian ensemble. We thus conjecture that measure synchronization is a generic feature of the dynamics of globally coupled Hamiltonian systems. The characterization of this kind of collective behavior in large systems and the study of the transition to measure synchronization is the subject of work in progress.

*Also at Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina.

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