Hard-Thermal-Loop Resummation of the Free Energy of a Hot Gluon Plasma

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We calculate the free energy of a hot gluon plasma to leading order in hard-thermal-loop perturbation theory. Effects associated with screening, gluon quasiparticles, and Landau damping are resummed to all orders. The ultraviolet divergences generated by the hard-thermal-loop propagator corrections can be canceled by a counterterm which depends on the thermal gluon mass. The deviation of the hard-thermal-loop free energy from lattice QCD results for $T > 2T_c$ has the correct sign and roughly the correct magnitude to be accounted for by next-to-leading order corrections.

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Relativistic heavy-ion collisions will soon allow the experimental study of hadronic matter at energy densities that should exceed that required to create a quark-gluon plasma. A quantitative understanding of the properties of a quark-gluon plasma is essential in order to determine whether it has been created. Because QCD, the gauge theory that describes strong interactions, is asymptotically free, its running coupling constant α_s becomes weak at sufficiently high temperatures. This would seem to make the task of understanding the high-temperature limit of hadronic matter relatively straightforward, because the problem can be attacked using perturbative methods. Unfortunately, the perturbative expansion in powers of α_s does not seem to be of any quantitative use even at temperatures that are orders of magnitude higher than those achievable in heavy-ion collisions.

The problem is evident in the free energy $\mathcal{F}$ of the quark-gluon plasma, whose weak-coupling expansion has been calculated through order $\alpha_s^{5/2}$ [1,2]. An optimist might hope to use perturbative methods at temperatures as low as 0.3 GeV, because the running coupling constant $\alpha_s(2\pi T)$ at the scale of the lowest Matsubara frequency is about $1/3$. However, the expansion in powers of $\alpha_s^{1/2}$ appears to converge only for extremely small values of α_s. For example, if $N_f = 6$, the $\alpha_s^{3/2}$ term is smaller than the α_s term only for $\alpha_s < 0.075$, which corresponds to a temperature greater than 10^3 GeV. At temperatures below 1 GeV, the corrections show no sign of converging, although the convergence can be somewhat improved by using Padé approximations [3]. It is clear that a reorganization of the perturbation series is essential if perturbative calculations are to be of any quantitative use at temperatures accessible in heavy-ion collisions.

The poor convergence of the perturbation series is puzzling, because lattice gauge theory calculations indicate that the free energy $\mathcal{F}$ of the quark-gluon plasma can be approximated by that of an ideal gas unless the temperature T is very close to the critical temperature T_c for the phase transition [4,5]. The deviation of $\mathcal{F}$ from the free energy of an ideal gas of massless quarks and gluons is less than about 25% if T is greater than $2T_c$. Furthermore, the lattice results can be described surprisingly well for all $T > T_c$ by an ideal gas of quark and gluon quasiparticles with temperature-dependent masses [6].

The large perturbative corrections seem to be related to plasma effects, such as the screening of interactions and the existence of quasiparticles, which arise from the momentum scale $\alpha_s^{1/2}T$. One possible solution is to use effective-field-theory methods to isolate the effects of the scale T [2,7], and then use a nonperturbative method to calculate the contributions from the lower momentum scales of order $\alpha_s^{1/2}T$ and smaller. The effective-field theory for the lower momentum scales is a gauge theory in three Euclidean dimensions, and it can be treated nonperturbatively using lattice-gauge-theory methods. Such an approach has been used by Kajantie *et al.* to calculate the Debye mass for thermal QCD [8]. One of the limitations of this approach is that it cannot be applied to the real-time processes that are the most promising signatures for a quark-gluon plasma.

An analogous convergence problem arises in the free energy of a massless scalar field theory with a ϕ^4 interaction, and several approaches to this problem have been proposed [9,10]. One of the most promising approaches is "screened perturbation theory" developed by Karsch, Patkós, and Petreczky [9], which involves a selective resummation of higher order terms in the perturbative expansion. This approach can be made systematic by using the framework of "optimized perturbation theory" [11]. A mass term proportional to ϕ^2 is added and subtracted from the Lagrangian, with the added term included nonperturbatively and the subtracted term treated as a perturbation. The renormalizability of the mass term guarantees that the new ultraviolet divergences generated by the mass term can be systematically removed by renormalization. When the free energy is calculated using screened perturbation theory, the convergence of successive approximations to the free energy is dramatically improved.

A straightforward application of screened perturbation theory to a gauge theory such as QCD is doomed to failure, because a local mass term for gluons is not gauge invariant. However, there is a way to incorporate plasma effects, including quasiparticle masses for gluons, into perturbation theory in a gauge-invariant way, and that

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is by using hard-thermal-loop (HTL) perturbation theory. This involves adding and subtracting HTL correction terms to the action [12], treating the quadratic parts of the added terms nonperturbatively and treating the remaining terms as interactions. The resulting effective propagators and vertices are complicated functions of the energies and momenta. The nonlocality of the HTL correction terms raises conceptual issues associated with renormalization, since the ultraviolet divergences they generate may not have a form that can be canceled by local counterterms.

In this Letter, we calculate the free energy of a hot gluon plasma explicitly to leading order in HTL perturbation theory. In spite of the complexity of the HTL propagators, their analytic properties can be used to make calculations tractable. Although complicated ultraviolet divergences arise in the calculation, many of them cancel. The remaining divergence is removed by a counterterm at the expense of introducing an arbitrary renormalization scale. With reasonable choices of the renormalization scales, the deviation of the HTL free energy from lattice QCD results for $T > 2T_c$ has the correct sign and roughly the correct magnitude to be accounted for by next-to-leading order corrections.

Our starting point is an expression for the free energy from the one-loop gluon diagram in which HTL corrections to the gluon propagator have been resummed. In the imaginary-time formalism, the renormalized free energy can be written as

$$\mathcal{F}_{\rm HTL} = 4(d-1)\sum\!\!\!\!\!\!\!\int [\omega_n^2 + k^2 + \Pi_T] + 4\sum\!\!\!\!\!\!\!\int [k^2 - \Pi_L] + \Delta\mathcal{F}\,, \tag{1}$$

where d is the number of spatial dimensions and $\Delta\mathcal{F}$ is a counterterm. The transverse and longitudinal HTL self-energy functions are

$$\Pi_T = -\frac{3}{2}\,m_g^2\,\frac{\omega_n^2}{k^2}\left[1 + \frac{\omega_n^2 + k^2}{2i\omega_n k}\log\frac{i\omega_n + k}{i\omega_n - k}\right], \tag{2}$$

$$\Pi_L = 3m_g^2\left[\frac{i\omega_n}{2k}\log\frac{i\omega_n + k}{i\omega_n - k} - 1\right], \tag{3}$$

where m_g is the gluon mass parameter. The sum-integrals in (1) represent $T\sum_n \mu^{3-d}\int d^dk/(2\pi)^d$, where the sum is over the Matsubara frequencies $\omega_n = 2\pi nT$. If we use dimensional regularization to regularize ultraviolet divergences, $\Delta\mathcal{F}$ cancels the poles in $d-3$ in the sum-integrals and μ is the minimal subtraction renormalization scale. If we set $\Pi_T = \Pi_L = 0$, the free energy (1) reduces to that of an ideal gas of massless gluons: $\mathcal{F}_{\rm ideal} = -(8\pi^2/45)T^4$.

Standard methods can be used to replace the sums over n in (1) by contour integrals in the energy $\omega = i\omega_n$. The integrands are weighted by the thermal factor $1/(e^{\beta\omega} - 1)$, and the contour encloses the branch cuts on the real ω axis. The arguments of the logarithms have branch cuts associated with Landau damping that extend from $-k$ to $+k$. The integrands also have logarithmic branch cuts that end at the points $\omega = \pm\omega_T(k)$ in the transverse term and at $\omega = \pm\omega_L(k)$ in the longitudinal term, where $\omega_T(k)$ and $\omega_L(k)$ are the quasiparticle dispersion relations for transverse gluons and longitudinal gluons (plasmons), respectively. These dispersion relations are the solutions to the following transcendental equations [13]:

$$\omega_T^2 = k^2 + \frac{3}{2}\,m_g^2\,\frac{\omega_T^2}{k^2}\left[1 - \frac{\omega_T^2 - k^2}{2\omega_T k}\log\frac{\omega_T + k}{\omega_T - k}\right], \tag{4}$$

$$0 = k^2 + 3m_g^2\left[1 - \frac{\omega_L}{2k}\log\frac{\omega_L + k}{\omega_L - k}\right]. \tag{5}$$

By collapsing the contours around the branch cuts, we can separate the integrals over ω into quasiparticle contributions and Landau-damping contributions. These individual contributions have severe ultraviolet divergences. The divergences can be isolated by subtracting expressions from the integrands that render the integrals finite in $d = 3$ and then evaluating the subtracted integrals analytically in d dimensions. If we impose a cutoff Λ on k and ω, there are power divergences proportional to Λ^4 and $m_g^2\Lambda^2$ and logarithmic divergences proportional to $m_g^2T^2\log\Lambda$, $m_g^4\log^2\Lambda$, and $m_g^4\log\Lambda$. The Λ^4 divergence is canceled by the usual renormalization of the vacuum energy density. The $m_g^4\log^2\Lambda$ divergences cancel between the quasiparticle and Landau-damping contributions to the transverse term. The cancellation can be traced to the fact that Π_T in (2) is analytic in the energy $\omega = i\omega_n$ at $\omega = \infty$. The temperature-dependent $m_g^2T^2\log\Lambda$ divergences cancel between the longitudinal and transverse terms. This cancellation follows from the identity $2\Pi_T - [(\omega_n^2 + k^2)/k^2]\Pi_L = 3m_g^2$. The remaining divergences arise from integration over large three-momentum and are canceled by the counterterm $\Delta\mathcal{F}$ in (1). In dimensional regularization, power divergences are set to zero and the logarithmic divergence appears as a pole in $d - 3$. In the minimal subtraction renormalization prescription, it is canceled by the counterterm $\Delta\mathcal{F} = -9m_g^4/[8\pi^2(d-3)]$.

Our final result for the free energy of the gluon plasma to leading order in HTL perturbation theory is

$$\mathcal{F}_{\rm HTL} = \frac{4T}{\pi^2}\int_0^\infty k^2\,dk\left[2\log(1 - e^{-\beta\omega_T}) + \log\frac{1 - e^{-\beta\omega_L}}{1 - e^{-\beta k}}\right] + \frac{4}{\pi^3}\int_0^\infty d\omega\,\frac{1}{e^{\beta\omega} - 1}\int_\omega^\infty k^2\,dk[\phi_L - 2\phi_T] + \frac{1}{2}\,m_g^2T^2 + \frac{9}{8\pi^2}\,m_g^4\left[\log\frac{m_g}{\mu_3} - 0.333\right], \tag{6}$$

where $\mu_3 = \sqrt{4\pi}\, e^{-\gamma/2}\mu$ is the renormalization scale associated with the modified minimal subtraction ($\overline{\text{MS}}$) renormalization prescription and γ is Euler's constant. The first term in (6) is the free energy of an ideal gas of transverse gluons with dispersion relation ω_T. The second term is the free energy of an ideal gas of plasmons with dispersion relation ω_L, with a subtraction that makes it vanish in the high-temperature limit $m_g \ll T$. The third term is a Landau-damping contribution that involves angles ϕ_L and ϕ_T defined by

$$\frac{3\pi}{4} m_g^2 \frac{\omega K^2}{k^3} \cot\phi_T = K^2 + \frac{3}{2} m_g^2 \frac{\omega^2}{k^2}\left[1 + \frac{K^2}{2k\omega} L\right], \quad (7)$$

$$\frac{3\pi}{2} m_g^2 \frac{\omega}{k} \cot\phi_L = k^2 + 3m_g^2\left[1 - \frac{\omega}{2k} L\right], \quad (8)$$

where $K^2 = k^2 - \omega^2$ and $L = \log[(k+\omega)/(k-\omega)]$. Both ϕ_T and ϕ_L vanish at the upper end point $k \to \infty$ of the integral over k. At the lower end point $k \to \omega$, ϕ_T vanishes and ϕ_L approaches π. The terms in (6) proportional to $m_g^2 T^2$ and m_g^4 come from the zero-point energies of the quasiparticles and from subtraction integrals. In the high-temperature limit $m_g \ll T$, $\mathcal{F}_{\text{HTL}}$ can be expanded in powers of m_g/T:

$$\mathcal{F}_{\text{HTL}} = \mathcal{F}_{\text{ideal}}\left[1 - \frac{45}{4} a + 30a^{3/2} + \frac{45}{8}\left(2\log\frac{\mu_3}{2\pi T} - 1.232\right)a^2 + \mathcal{O}(a^3)\right], \quad (9)$$

where $a = 3m_g^2/(4\pi^2 T^2)$. Only integer powers of a appear beyond the $a^{3/2}$ term. In the limit $T \to 0$ with m_g fixed, $\mathcal{F}_{\text{HTL}}$ is proportional to m_g^4:

$$\mathcal{F}_{\text{HTL}} \to \frac{9}{8\pi^2}\left(\log\frac{m_g}{\mu_3} - 0.333\right)m_g^4. \quad (10)$$

This low-temperature limit is sensitive to the value of μ_3. In particular, the coefficient of m_g^4 in (10) changes sign at $\mu_3 = 0.717 m_g$.

The free energy of a quark-gluon plasma in the high-temperature limit has been calculated in a weak-coupling expansion through order $\alpha_s^{5/2}$ [1,2]. The result for a pure gluon plasma with $N_c = 3$ is

$$\mathcal{F}_{\text{QCD}} = \mathcal{F}_{\text{ideal}}\left[1 - \frac{15}{4} a + 30a^{3/2} + \frac{135}{2}(\log a + 3.51)a^2 - 799.2a^{5/2} + \mathcal{O}(a^3 \log a)\right], \quad (11)$$

where $a = \alpha_s(2\pi T)/\pi$. In the limit $\alpha_s \to 0$, the gluon mass parameter m_g is given by $m_g^2 = (4\pi/3)\alpha_s T^2$. The expansion parameters a in (9) and (11) therefore coincide in this limit. The order-$a^{3/2}$ terms in these expansions are identical, because HTL resummation includes the leading effects associated with Debye screening. Note that HTL resummation overincludes the order-a correction by a factor of 3. The remaining order-a corrections would appear at next-to-leading order in HTL perturbation theory. The order-a correction in $\mathcal{F}_{\text{HTL}}$ together with the corrections that are higher order in a combine to give a total correction that is negative, in spite of the large positive contribution from the $a^{3/2}$ term.

In this leading order calculation, T, m_g, and μ_3 all appear as independent parameters. The parameters m_g and μ_3 should be chosen as functions of T and α_s so as to avoid large higher order corrections in HTL perturbation theory. At asymptotically large temperatures, the fractional correction to $\mathcal{F}_{\text{ideal}}$ from the next-to-leading order diagrams must reduce to $+(15/2)\alpha_s/\pi$ in order to agree with $\mathcal{F}_{\text{QCD}}$ up to corrections of order α_s^2. This will require setting the thermal gluon mass parameter to

$$m_g^2(T) = \frac{4\pi}{3}\alpha_s(\mu_4)T^2, \quad (12)$$

with a renormalization scale μ_4 of order T. A reasonable choice is $\mu_4 = 2\pi T$, the Euclidean energy of the lowest Matsubara mode. The logarithmic divergences associated with the three-dimensional renormalization scale μ_3 will be cut off by higher order corrections at a scale of either m_g or T. If we choose μ_3 to be of order m_g, the $a^2\log(\mu_3/T)$ term in (9) will reproduce a fraction of the $\alpha_s^2 \log\alpha_s$ term in $\mathcal{F}_{\text{QCD}}$. A reasonable choice is $\mu_3 = \sqrt{3}\, m_g(T)$, which is the Debye screening mass.

In Fig. 1, we compare various approximations to the free energy of a gluon plasma to the lattice results for pure-glue QCD from Boyd *et al.* [4]. The unshaded bands are the ranges of the perturbative expansions of the QCD free energy when the renormalization scale is varied by a factor of 2 from the central value $\mu_4 = 2\pi T$. The four bands correspond to $\mathcal{F}_{\text{QCD}}$ in (11) truncated after the α_s, $\alpha_s^{3/2}$, α_s^2, and $\alpha_s^{5/2}$ terms, respectively. We use a running coupling constant that runs according to the two-loop beta function: $\alpha_s(\mu_4) = (4\pi)/(11\bar{L})[1 - (102/121)\log(\bar{L})/\bar{L}]$, where $\bar{L} = \log(\mu_4^2/\Lambda_{\overline{\text{MS}}}^2)$. The parameter $\Lambda_{\overline{\text{MS}}}$ is related to the critical temperature T_c by $T_c = 1.03\Lambda_{\overline{\text{MS}}}$ [4]. The poor convergence properties of the perturbative expansion and the strong dependence on the renormalization scale are evident in Fig. 1. The shaded region in Fig. 1 is the range of the HTL free energy $\mathcal{F}_{\text{HTL}}$ when the renormalization scales are varied by a factor of 2 from the central values $\mu_4 = 2\pi T$ and $\mu_3 = \sqrt{3}\, m_g(T)$. For these choices of μ_3 and μ_4, $\mathcal{F}_{\text{HTL}}/T^4$ is a slowly increasing function of T. This feature follows from the fact that $m_g(T)$ is approximately linear in T, with deviations from linearity coming only from the running of the coupling constant. If $m_g(T)$ was exactly linear in T, $\mathcal{F}_{\text{HTL}}/T^4$ would be independent of T. With our choices of μ_3 and μ_4, the HTL free energy lies significantly below the lattice results for $T > 2T_c$.

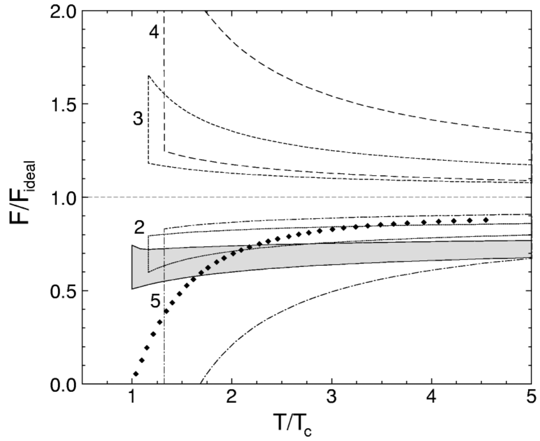

FIG. 1. The free energy of a hot gluon gas normalized to that of an ideal gas of gluons as a function of T/T_c. The black diamonds are the lattice results for pure-glue QCD from Ref. [4]. The bands enclosed by the curves labeled 2, 3, 4, and 5 are the QCD free energy $\mathcal{F}_{\rm QCD}$ truncated after the α_s, $\alpha_s^{3/2}$, α_s^2, and $\alpha_s^{5/2}$ terms, respectively. The bands correspond to varying μ_4 by a factor of 2 from the central value $2\pi T$. The shaded region is the HTL free energy $\mathcal{F}_{\rm HTL}$ with $m_g^2 = (4\pi/3)\alpha_s(\mu_4)T^4$. The region corresponds to varying the renormalization scales by a factor of 2 from the central values $\mu_3 = \sqrt{3}\, m_g$ and $\mu_4 = 2\pi T$.

This should not be of great concern, because the next-to-leading order correction in HTL perturbation theory will give a fractional correction to $\mathcal{F}_{\rm ideal}$ that approaches $+(15/2)\alpha_s(\mu_4)/\pi$ at asymptotic temperatures. It has the correct sign and roughly the correct magnitude to decrease the discrepancy with the lattice QCD results at the highest values of T. With the inclusion of the next-to-leading order correction, the error at asymptotic temperatures will fall as $\alpha_s^2 \log\alpha_s$. If the next-to-next-to-leading order correction was also included, the error would decrease to order $\alpha_s^3 \log\alpha_s$. Because of the magnetic mass problem, the error can be decreased below order α_s^3 only by using nonperturbative methods.

We have proposed HTL perturbation theory as a resummation prescription for the large perturbative corrections associated with screening, quasiparticles, and Landau damping. The free theory around which we are perturbing is similar to the phenomenological quasiparticle models, but the effects of interactions between the quasiparticles can be systematically calculated. This approach can be applied to the real-time processes that may serve as signatures for the quark-gluon plasma. We have demonstrated that HTL perturbation theory is tractable by calculating the leading term in the free energy of a pure gluon plasma. With reasonable choices of the renormalization scales, the deviation of the hard-thermal-loop free energy from lattice QCD results for $T > 2T_c$ has the correct sign and roughly the correct magnitude to be accounted for by next-to-leading order corrections. A challenging problem is to extend the calculation of the free energy to next-to-leading order in HTL perturbation theory. If the next-to-leading order correction proves to be small for temperatures within an order of magnitude of T_c, we may finally have a perturbative framework that will allow quantitative calculations of the properties of a quark-gluon plasma at experimentally accessible temperatures.

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