

Kinetic Eigenmodes and Discrete Spectrum of Plasma Oscillations in a Weakly Collisional Plasma

C. S. Ng, A. Bhattacharjee, and F. Skiff

Department of Physics and Astronomy, The University of Iowa, Iowa City, Iowa 52242

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The damping of plasma oscillations in a weakly collisional plasma is revisited using a Fokker-Planck collision operator. It is shown that the Case–Van Kampen continuous spectrum is eliminated in the limit of zero collision frequency and replaced by a discrete spectrum. The Landau-damped solutions are recovered in this limit, but as true eigenmodes of the weakly collisional system. For small but nonzero collision frequency, the spectra and eigenmodes are qualitatively different from their counterparts in the collisionless theory. These results are consistent with recent experimental findings.

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Landau damping of plasma oscillations in a collisionless plasma is one of the most fundamental and widely used concepts in plasma physics. Although Landau's classic paper [1] is the standard point of departure for discussions of kinetic stability theory in most textbooks, it raises some vexing physical questions. Why should plasma oscillations damp in a collisionless plasma in which the underlying dynamics is time reversible and nondissipative? If the asymptotic time dependence of linear perturbations in a collisionless plasma is written in the form $\exp(-i\omega t)$, the Landau theory predicts damped plasma oscillations with $\text{Im}\omega < 0$ for monotonic distribution functions. But unlike the unstable solutions for nonmonotonic distribution functions, the damped solutions are not eigenmodes. What is the physical reason for this strange asymmetry? If the plasma supports unstable eigenmodes with $\text{Im}\omega > 0$, why does it not support stable eigenmodes with $\text{Im}\omega < 0$ when the distribution function is monotonic?

These questions have been answered by the work of Van Kampen [2], Case [3], and Dawson [4] within the framework of the collisionless theory. A key to understanding Landau damping is phase mixing, made possible by the presence of a continuous spectrum which lies on the real axis ($\text{Im}\omega = 0$) of the complex- ω plane. This continuous spectrum is associated with a complete but singular set of eigenmodes, known as the Case–Van Kampen modes (discussed in textbooks, e.g., [5–7]). It takes very special, that is, singular initial conditions to excite isolated Case–Van Kampen modes. In most situations of physical interest where the initial conditions are smooth, a broad and continuous spectrum of Case–Van Kampen modes is excited. The Landau-damped waves are not eigenmodes but are remnants, in the long-time limit, after a continuous and complete set of singular eigenmodes, each one of which is purely oscillatory, have interfered destructively (in the sense of the Riemann-Lebesgue theorem).

How is this widely accepted physical picture of Landau damping modified if collisions are introduced? Lenard and Bernstein [8] considered the problem using an operator of the Fokker-Planck-type [9]. They obtained an

exact analytic solution with a dispersion relation which formally reduces to that of Landau in the limit of zero collisions, but they did not discuss the nature of the spectrum or the effect of collisions on the Van Kampen eigenmodes. Su and Oberman [10] (and Karpman [11]) claimed that plasma wave echoes [12] (and the ballistic response) which owe their existence to the intrinsic time reversibility of the Vlasov equation, should decay very rapidly due to the presence of collisions. Specifically, using the Lenard-Bernstein collision operator, they predicted that spatial echoes should decay as $\exp[-(\nu/\omega_p \lambda_D^3)x^3]$ and temporal echoes as $\exp[-\nu\omega_p^2 t^3]$, where ν is the collision frequency, ω_p is the plasma frequency, and λ_D is the Debye length. Su and Oberman made the crucial assumption, repeated subsequently in textbooks [6], that in the presence of collisions an eigenfunction can always be found for any real value of ω . In other words, they assumed implicitly that the continuous spectrum remains intact in the presence of collisions.

In a recent experiment [13] involving a weakly collisional stable plasma, the measured decay rate for the least damped electrostatic ion perturbations was found to be substantially weaker and scaled quite differently than predicted by the Su-Oberman theory. Skiff *et al.* [13] suggested that collisions change the spectrum of plasma excitations and presented experimental and numerical evidence that the electrostatic ion wave spectrum is discrete.

The evidence presented in [13] motivates us to revisit the classical problem of Landau damping of plasma oscillations, including collisions in the theory. We demonstrate that the Case–Van Kampen continuous spectrum is eliminated for the Lenard-Bernstein collision operator, even in the limit of zero collision frequency ($\nu \rightarrow 0$). The eigenmodes are qualitatively different from Case–Van Kampen modes in the $\nu \rightarrow 0$ limit. We do recover the Landau-damped solutions in the $\nu \rightarrow 0$ limit, but in contrast to their character in the collisionless theory ($\nu = 0$), they are discrete eigenmodes of the weakly collisional system. Furthermore, in this limit, we find new damped eigenmodes, not obtained by the Landau analysis. For

nonzero values of ν , the discrete spectrum deviates substantially from that obtained by the Landau analysis.

We concur with the result [13] that the Su-Oberman theory of collisional damping of plasma echoes is in error. Su and Oberman observed correctly that for small values of ν the Fokker-Planck collision operator involves singular perturbations on the collisionless problem, but erred by assuming implicitly that the Case-Van Kampen continuous spectrum is preserved in the presence of collisions.

In contrast with [13] which dealt with electrostatic ion perturbations, we assume that the ion distribution is unperturbed, and focus on electrostatic electron perturbations [1-3,8,10]. However, the main conclusions of this paper remain the same for the ion problem, despite some technical differences. We begin with the one-dimensional linearized equations for the electron distribution function $f(x, t, v)$, coupled with the self-consistent Coulomb's law for the electric field,

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - \frac{e}{m} \frac{\partial f_0}{\partial v} E = \nu \frac{\partial}{\partial v} \left(v f + v_0^2 \frac{\partial f}{\partial v} \right), \quad (1)$$

$$\frac{\partial E}{\partial x} = -4\pi e \int_{-\infty}^{\infty} dv f(x, t, v). \quad (2)$$

The collision frequency ν is a constant, and $-e$, m , and v_0 are, respectively, the charge, mass, and thermal speed of the electron. The function f_0 is the equilibrium Maxwellian distribution. The right-hand side of (1) is the linearized collision operator used in [8,10,11]. Unlike the full Fokker-Planck collision term representing electron-electron as well as electron-ion collisions [14], the form (1) neglects electron-ion collisions and assumes, furthermore, that the electron-electron collision frequency ν does not fall off with increasing velocity. However, the form (1) does conserve particles, satisfies the H theorem, and preserves the diffusive character of the full Fokker-Planck operator in velocity space.

We seek solutions of the form $f(x, t, v) = \tilde{f}(k, \omega, v) \exp[i(kx - \omega t)]$. In what follows, we consider two problems at once: (i) the temporal evolution problem, for which k is real and ω is complex, and (ii) the spatial evolution problem, for which ω is real and k is complex. The temporal evolution problem is most commonly discussed in textbooks, but the spatial evolution problem is of greater relevance for experiments [13]. We define a slightly different set of dimensionless variables in cases (i) and (ii). Combining Eqs. (1) and (2), we obtain

$$(u - \Omega)g(u) - \eta(u) \int_{-\infty}^{\infty} g(u') du' = -i\mu \frac{\partial}{\partial u} \left(ug + \frac{1}{2} \frac{\partial g}{\partial u} \right), \quad (3a)$$

for case (i), where $u \equiv v/(\sqrt{2}v_0)$, $\Omega \equiv \omega/(\sqrt{2}kv_0)$, $g \equiv \sqrt{2}v_0\tilde{f}/n_0$, $g_0 \equiv \exp[-u^2]/\pi^{1/2}$, n_0 is the equilibrium electron density, $\eta(u) \equiv \alpha(\partial g_0/\partial u)/2$, $\alpha \equiv \omega_p^2/(k^2v_0^2) = 4\pi n_0 e^2/(mk^2v_0^2)$, and $\mu \equiv \nu/(\sqrt{2}kv_0)$. For case (ii), we obtain

$$\left(u - \frac{1}{\kappa} \right) g(u) - \frac{\eta(u)}{\kappa^2} \int_{-\infty}^{\infty} g(u') du' = -i \frac{\mu}{\kappa} \frac{\partial}{\partial u} \left(ug + \frac{1}{2} \frac{\partial g}{\partial u} \right), \quad (3b)$$

where $\kappa \equiv \sqrt{2}kv_0/\omega$, and we redefine $\alpha \equiv 2\omega_p^2/\omega^2$, $\mu \equiv \nu/\omega$, keeping all other definitions the same as for case (i).

For the collisionless problem ($\mu = 0$), it has been shown by Van Kampen [2] and Case [3] that any real Ω is an eigenvalue with a singular eigenfunction given by

$$g_{\Omega}(u) = P \left[\frac{\eta(u)}{u - \Omega} \right] + \delta(u - \Omega) \left[1 - P \int_{-\infty}^{\infty} \frac{\eta(u')}{u' - \Omega} du' \right], \quad (4)$$

where P denotes the principal part. In other words, the spectrum of eigenvalues is a continuum that lies on the real axis in the complex Ω plane. On the other hand, Landau [1] showed that the plasma oscillations are damped according to the relation

$$D(\Omega_n) \equiv 1 - \int \frac{\eta(u)}{u - \Omega_n} du = 0, \quad (5)$$

with an infinite number of discrete roots Ω_n , each with a negative imaginary part. A smooth initial condition $g(u)$ can be represented by the linear superposition of a complete set of singular Case-Van Kampen eigenmodes

$g_{\Omega}(u)$ which, in the limit $t \rightarrow \infty$, decay with the damping rate specified by (5). The decaying solution is not an eigenmode, but it is what remains in the long-time limit after a continuous spectrum of purely oscillatory eigenmodes have phase mixed. (Strictly speaking, in the $t \rightarrow \infty$ limit, the collisionless theory is fundamentally problematic, because the velocity gradient of f becomes singular unless collisions are invoked to smoothen out the singularity.)

In Fig. 1, we indicate by crosses (\times) the Landau roots of the spatial evolution problem in the complex κ plane for $\alpha = 1.6$, obtained by solving the dispersion relation $1 + \alpha[1 + Z(1/\kappa)/\kappa]/\kappa^2 = 0$, where Z is the plasma dispersion function. The Landau roots for the temporal evolution problem are qualitatively similar if plotted in the complex Ω^{-1} plane. For both problems (i) and (ii), there are roots with progressively smaller spacing near the origin which lie asymptotically on the straight line through the origin and the point (1, 1).

We now discuss how the collisionless ($\mu = 0$) results change in the collisionless limit $\mu \rightarrow 0$, since this represents a singular perturbation. Following [13] and

[15], we solve (3a) by introducing an expansion based on a complete set of normalized Hermite polynomials,

$g = \sum_{n=0}^{\infty} a_n H_n(u) \exp(-u^2)$, so that (3a) reduces to the matrix eigenvalue problem,

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 1 + \alpha & 0 & \sqrt{2} & 0 & 0 & \dots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & \dots \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} & \dots \\ 0 & 0 & 0 & \sqrt{4} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \end{pmatrix} = \sqrt{2} \begin{pmatrix} \Omega a_0 \\ (\Omega + i\mu)a_1 \\ (\Omega + 2i\mu)a_2 \\ (\Omega + 3i\mu)a_3 \\ (\Omega + 4i\mu)a_4 \\ \vdots \end{pmatrix}, \tag{6}$$

for the infinite-dimensional vector a_n . In principle, the problem can be solved by keeping up to n terms and solving numerically the resultant $n \times n$ matrix equation to determine the eigenvalues [13,15]. However, because the coefficient of μ increases linearly with n on the right-hand side of (6), we need to keep an increasingly large number of terms in the limit $\mu \rightarrow 0$. This requirement imposes a severe constraint on the minimum value of μ for which a direct numerical solution of the matrix equation (6) can be obtained. It is less constraining to rewrite (6) as a recurrence relation

$$\begin{aligned} a_1 &= \sqrt{2} \Omega a_0, \\ a_2 &= [(\Omega + i\mu)a_1 - (1 + \alpha)a_0/\sqrt{2}], \\ a_{n+1} &= \sqrt{\frac{2}{n+1}} \left[(\Omega + in\mu)a_n - \sqrt{\frac{n}{2}} a_{n-1} \right] \end{aligned} \tag{7}$$

for $n \geq 2$.

It can be shown easily that

$$\frac{a_{n+1}}{a_n} \xrightarrow{n \rightarrow \infty} \begin{cases} i\mu\sqrt{2n} & \text{("large")} \\ 1/(i\mu\sqrt{2n}) & \text{("small")} \end{cases}. \tag{8}$$

Since a physically acceptable eigenfunction must satisfy the condition $a_n \rightarrow 0$ as $n \rightarrow \infty$, the eigenvalue problem reduces to a search for Ω such that the iteration of (7)

converges on the "small" solution. To avoid numerical instability during iteration, we start out with a large n and iterate (7) backwards. This ensures that the backward series stays close to the small solution, but the backward series will not necessarily match the forward series unless Ω is an eigenvalue. After the eigenvalue is determined by matching the two series to a specified level of accuracy, the eigenfunction is simply obtained by substituting the values of a_n in the Hermite expansion.

Figure 1 shows numerical results for the spatial evolution problem for $\alpha = 1.6$. In addition to the Landau roots ($\mu = 0$) which are marked by crosses (\times), we plot the weakly collisional roots for three other values of μ : $\mu = 0.1$, denoted by diamonds (\diamond); $\mu = 0.05$, denoted by stars ($*$); and $\mu = 0.025$, denoted by plusses ($+$). (The last set of points was resolved using quadruple precision.)

The Landau root for the least-damped solution is indicated by a cross very close to the real axis, near $\kappa_r \approx 0.33$. The weakly collisional, least-damped eigenvalue tends to the least-damped Landau solution as μ decreases. The numerical data illustrating this behavior are shown in Table I. We see that the correction to the imaginary part of the least-damped Landau root is approximately given by $i\mu$. However, as shown in Fig. 1, other damped Landau roots suffer more drastic changes due to the presence of collisions, not described by the simple $(\kappa + i\mu)$ rule. In Table II, we present the numerical data on the least-damped eigenvalue of the temporal evolution problem for $\alpha = 9$. Again, the least-damped eigenvalue tends to the least-damped Landau solution as μ decreases, with a correction that is approximately given by $-i\mu$.

Lenard and Bernstein derived a dispersion relation ($D_{LB} = 0$) for the collisional problem which tends to the Landau relation ($D_L = 0$) in the limit $\mu \rightarrow 0$. This motivates us to write $D_{LB} = D_L + \mu D_1 + \mu^2 D_2 + \dots$

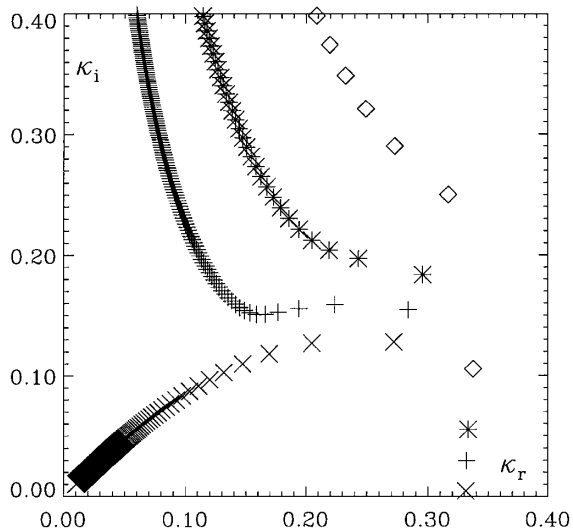


FIG. 1. Eigenvalues on the complex κ plane for the spatial evolution problem with $\alpha = 1.6$. (\diamond) $\mu = 0.1$; ($*$) $\mu = 0.05$; ($+$) $\mu = 0.025$; (\times) Landau roots ($\mu = 0$).

TABLE I. Numerical values of the least-damped eigenvalue κ for different values of μ with $\alpha = 1.6$ for the spatial evolution problem.

μ	κ_r	κ_i
0.1	0.337 065 2	0.105 687 1
0.01	0.331 301 3	0.014 547 5
0.001	0.331 130 0	0.005 465 0
0.0001	0.331 117 3	0.004 558 8
Landau root ($\mu = 0$)	0.331 115 9	0.004 458 2

TABLE II. Numerical values of the least-damped eigenvalue Ω for different μ with $\alpha = 9$ for the temporal evolution problem.

μ	Ω_r	Ω_i
0.1	2.517 732 3	-0.127 010 1
0.01	2.542 284 6	-0.062 245 8
0.001	2.545 516 7	-0.055 623 7
0.0001	2.545 785 5	-0.054 960 1
0.00001	2.545 812 4	-0.054 893 7
Landau root ($\mu = 0$)	2.545 815 4	-0.054 886 4

However, the Lenard-Bernstein dispersion relation, $D_{LB} = 0$ admits additional roots, not included in the Landau relation, $D_L = 0$. It is easy to see that in the limit $\mu \rightarrow 0$, these additional roots (for which $D_L \neq 0$) must tend to poles of the Lenard-Bernstein dispersion relation so that the finite value of D_L can be canceled by other terms in the expansion of D_{LB} in powers of μ . For the spatial and temporal evolution problems, these poles are located, respectively, at $\Omega = -i[n\mu + 1/(2\mu)]$ and $\kappa^2 = 2(i\mu - n\mu^2)$, where $n = 0, 1, 2, 3, \dots$. It follows that for the spatial evolution problem $\kappa_r^2 - \kappa_i^2 = -2n\mu^2$, and $\kappa_r \kappa_i = \mu$. Hence as $\mu \rightarrow 0$, these additional eigenvalues asymptote to the imaginary κ axis and there exists a limiting continuum (the imaginary- κ axis) which becomes discrete for any nonzero μ . This continuum should be distinguished from the Case-Van Kampen continuum (the real- κ axis) which is eliminated even for extremely weak collisions. For the temporal evolution problem, the additional roots all tend to $-i\infty$ and represent modes that are damped very quickly.

In Fig. 2, we plot the eigenfunction (solid line) of the least-damped mode with eigenvalue Ω_0 (numerical values in Table II) of the temporal evolution problem with $\alpha = 9$. We compare the numerical eigenfunction to the analytical exterior region solution $\eta(u)/(u - \Omega_0)$. We see that the numerical and analytical solutions agree very well in the exterior region of u centered around Ω_{0r} . Figure 2 might suggest that the boundary layer, where the eigenfunction departs from the exterior region solution, shrinks as μ decreases, but this is not what actually occurs. In fact, we find numerically that as $\mu \rightarrow 0$, the boundary layer tends to a fixed width (of the order of $|\Omega_{0i}|$) while the eigenfunction within the layer becomes increasingly singular.

The results presented above call for the development of a new asymptotic theory for the damping of plasma oscillations in weakly collisional plasmas. For the Lenard-Bernstein collision operator, we conjecture that the class of discrete eigenmodes, some of which we have calculated numerically, is complete, but we do not have a formal proof yet. It is possible that a more realistic collision operator in which the collision frequency has velocity dependence [14] may support a continuous spectrum (distinct from the Case-Van Kampen continuum) in addition to the discrete spectrum discussed above. Also, the inclusion of electron-ion collisions is likely to alter the actual

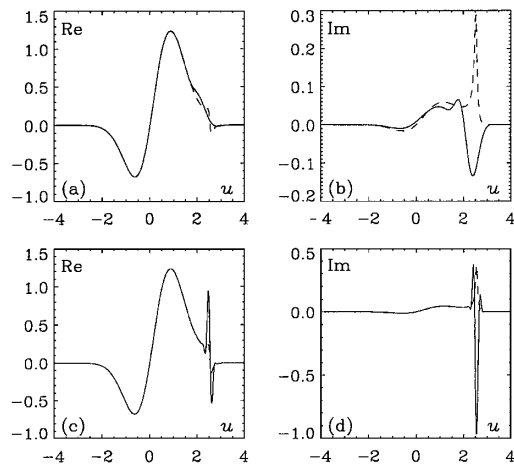


FIG. 2. Solid curves: eigenfunctions corresponding to the least-damped eigenvalue Ω_0 for the temporal evolution problem with $\alpha = 9$. Dashed curves: the function $\eta(u)/(u - \Omega_0)$. (a) and (b): Real and imaginary parts of the eigenfunction, respectively, for $\mu = 0.025$. (c) and (d): Real and imaginary parts of the eigenfunction, respectively, for $\mu = 0.000390625$.

damping rates. However, we believe that our results on the destruction of the Case-Van Kampen continuum and its replacement by a discrete spectrum will continue to be essential features of the weakly collisional theory, even if a more realistic collision operator is used.

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