Surface Wave Scattering by a Vertical Vortex and the Symmetry of the Aharonov-Bohm Wave Function

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Surface waves interacting with a vertical vortex are studied both analytically and experimentally. There are similarities as well as differences with the Aharonov-Bohm effect: among the former, a dislocation in the wave fronts that is proportional to the vortex circulation, and among the latter, a significant change in the symmetry of the scattered wave.

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Ever since the appearance of the landmark paper by Aharonov and Bohm [1], the fact that the dynamics of a charged quantum particle is affected by the vector potential even when the magnetic field vanishes has been a source of inspiration and applications in widely different fields of physics. Berry [2] showed that this effect was embedded in a more general geometrical framework ("Berry's phase"), a fact that has also had wide reaching consequences [3].

In a different development, Berry *et al.* [4] cleared up many issues arising when electrons interact with the vector potential generated by a magnetic field confined to a cylinder into which the electrons cannot penetrate, and showed that the global aspects of the Aharonov-Bohm effect had a water wave analog. They noted that there appears a dislocation on the electron wave fronts that cannot be observed, but that the analogous effect on water waves incident on an irrotational vortex can. The global aspect of geometric phases, involving integrals around closed paths in suitably defined spaces, is what gives them their generality. Each specific problem, however, has its own peculiarities. In this Letter, we report results of a detailed study of the interaction of a water wave with a vortex.

The study whose results we report herein has an additional motivation: coherent structures such as vortex filaments are prominent features of turbulent flows [5], and vortex stretching is believed to be a major ingredient in the cascade of energy from large to small scales that is characteristic of those flows. However, due to the paucity of nonintrusive experimental methods, the characterization of vortical structures is still poor. Recently, it has become apparent that it is possible to obtain valuable information about vorticity fields by looking at their interaction with an ultrasonic wave [6]. A full visualization of an acoustic wave is, however, a difficult problem, and it is in order to get additional insight into the wave-vortex interaction that we have undertaken this study of the more readily visualizable surface waves. This problem is also of interest on its own right, in order to characterize surface flows in rivers, lakes, and seas [7].

Here we report on an experimental study of the scattering of a capillary water wave by a single vertical vortex. Capillary waves are dispersive while acoustic waves, closely analogous to shallow water waves, are not. In order to bridge the gap between the two cases, and also because it lends itself more easily to comparison with the quantum mechanical case, we have centered our theoretical study on surface waves over water of moderate undisturbed depth h, (constant) density ρ, and surface tension τ. In this case the appropriate dynamical variable $\psi(\vec{r}, t)$ is the surface elevation of water over its undisturbed level as a function of position in two space dimensions and time, and it obeys the differential equation [8]

$$gh\nabla^2\psi + \left(\frac{1}{3}\,gh^3 - \frac{\tau h}{\rho}\right)\nabla^4\psi - D_t^2\psi = 0\,, \qquad (1)$$

where g is the acceleration due to gravity, and $D_t \equiv \partial_t + \vec{U}\cdot\nabla$. The velocity $\vec{U}$ is associated with a vertical vortex. The shallow water limit obtains when the scale for space variations is much larger than the fluid depth h so that fourth order derivatives in (1) can be neglected. The wave equation that results also describes acoustic waves if $\sqrt{gh}$ is identified with the speed of sound and ψ with acoustic pressure. The interesting situation for present purposes is when the vorticity $\nabla\times\vec{U}$ vanishes outside a circle, say, of radius a. To fix ideas, assume that inside the fluid undergoes rigid body rotation.

An electron in the presence of a magnetic field with vector potential $\vec{A}$ is described by a wave function ψ that obeys Schrödinger's equation:

$$[-i\hbar\nabla - q\vec{A}(\vec{r})]^2\psi(\vec{r}) = \hbar^2k^2\psi(\vec{r})\,. \qquad (2)$$

Berry *et al.* [4] studied in detail the two dimensional situation when the magnetic field associated with $\vec{A}$ vanishes outside a circle of radius a and the wave function ψ cannot penetrate inside. They concluded that when a plane wave is incident on the circle, the solution to (2) consists of two portions: one is a dislocated modification of the incident plane wave, and another is a scattered wave. This separation is, however, meaningless in the

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forward direction. The incident wave is dislocated by an amount $\alpha \equiv q\Phi/\hbar$ where Φ is the magnetic flux associated with the magnetic field contained within the circle of radius a. On the basis of the dispersion relation obeyed by water waves, they concluded that a similar separation would occur in the case of a surface water wave incident on a vertical vortex and that in this case the amount of dislocation would be given by $\alpha = \Gamma/\lambda c_g$, where Γ is the circulation associated with the vortex, λ is the wavelength of the incident wave, and c_g the group velocity of surface waves. By analogy with crystal dislocations, we shall call α the (dimensionless) Burger's vector of the wave.

It is possible to analytically solve Eq. (1) and the solution bears out the global considerations of the previous paragraph. There is indeed a dislocation with magnitude given by the reasoning of Berry *et al.* [4]. However, there is more: First, Eqs. (1) and (2), although of a similar structure, are different. Second, water waves penetrate inside the vortex, while electrons do not necessarily penetrate inside a solenoid. These differences do not affect the fact that in both cases there is a dislocated wave, but they do affect significantly the scattered wave. The most striking difference is that in the quantum mechanical case with impenetrable boundary conditions the scattered wave is symmetric (see Fig. 1, upper panel). As we will see below, in the water wave case the scattered wave is asymmetric. This difference is due to the fact that water wave fronts are rotated by a finite amount *inside* the vortex. This rotation also occurs when electrons interact with a magnetic field. Indeed, solving Eq. (2) allowing the electron to penetrate inside the solenoid leads to an asymmetric scattered wave [9] (see Fig. 1, lower panel). For impenetrable boundary conditions, the wave function is assumed to vanish inside the solenoid, whereas for penetrable boundary conditions we require the wave function to be finite at $r = 0$, and continuity of the function and its first derivative at the solenoid (vortex core) boundary.

To observe the surface wave case, we have performed an experiment in which a plane fronted wave is excited by moving horizontally a rigid dipper at the surface of a water tank 10 cm deep, at frequencies and amplitudes that range from 5 to 40 Hz and from 0.5 to 0.1 cm, respectively [10]. The vortex forms spontaneously when letting the water come out off the bottom of the tank through a hole of 6 mm diameter. The vortex circulation is enhanced by rotating, around its principal axis, a disk of 15 cm diameter located close to the bottom. The fluid within the vortex core performs rigid body rotation, whereas the tangential velocity far from the vortex core decays approximately as $1/r$. The vortex core a is estimated to be about 4 mm. In order to visualize the scattered wave, the free surface of the fluid was illuminated from above with a parallel beam. A semitransparent mirror located at 45° with respect to the horizontal deviates an horizontal incident beam

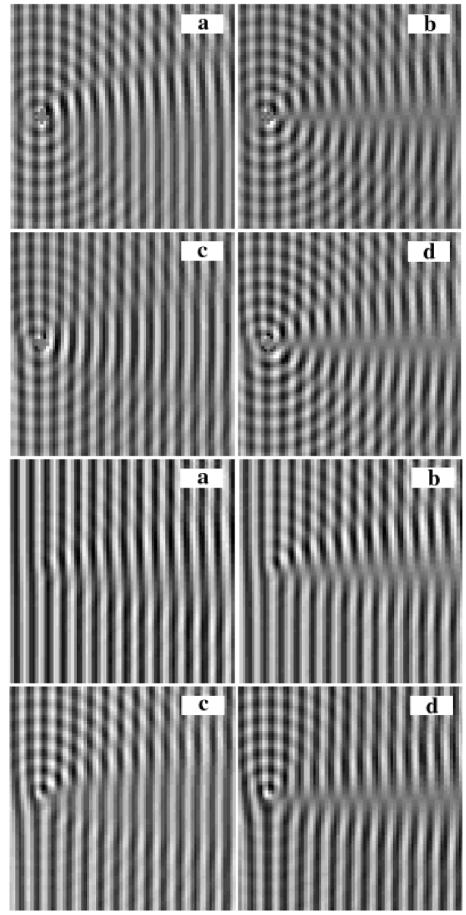

FIG. 1. Upper panel: Electron wave function for the scattering by a magnetic field wholly contained within an impenetrable solenoid of radius $a = 0.36\lambda$ for $\alpha = 0.2$ (a), 0.5 (b), 1.0 (c), and 1.5 (d). Lower panel: same as upper panel but electron is now able to penetrate the solenoid. Note the change in the symmetry of the scattered wave. In all figures the electron propagates from left to right and the vector potential is positive. In both the penetrable and impenetrable case there is a dislocated wave front with Burgers vector proportional to the magnetic flux.

providing homogeneous and perpendicular lighting. The light reflected on the wavy surface of water crosses the mirror and forms caustic lines on a horizontal screen made of diffusing glass and located just above the mirror.

Figure 2 shows the solution of Eq. (1) and the experimental visualization of surface waves scattered by a vertical vortex. These results can be compared to the solution of Eq. (2) with impenetrable as well as penetrable boundary conditions (see Fig. 1). All four cases exhibit wave fronts with dislocations that increase with the magnetic flux, in the quantum mechanical case, and with the vortex circulation in the classical case. The quantum mechanical case has a scattered wave that is up-down symmetric when the cylinder is impenetrable, but this symmetry no longer holds when the electrons can penetrate inside the cylinder. This asymmetry is also present in the water

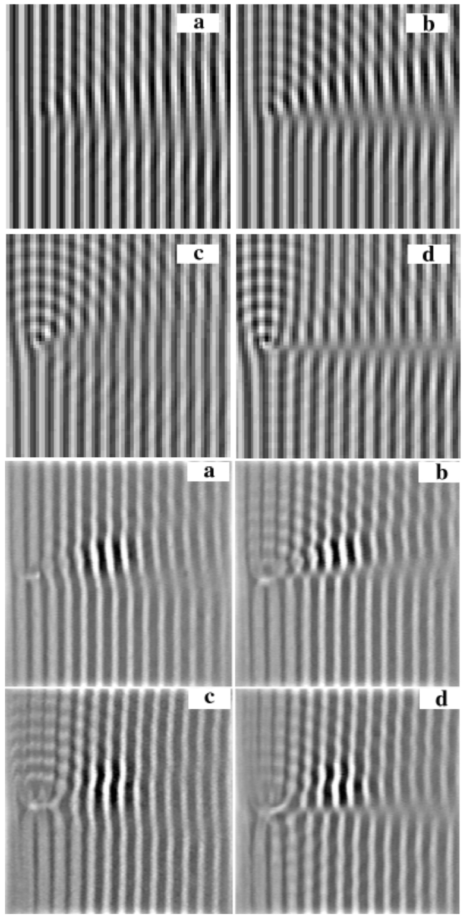

FIG. 2. Surface elevation for water waves scattered by a vertical vortex. Theory, upper panel, and experiment, lower panel. Water wave results correspond to $f = 20$ Hz, $\lambda = 1.0$ cm, $c_g = 22.2$ cm/s, $a \sim 0.4\lambda$, and varying Γ. The incident wave propagates from left to right and the fluid motion is counterclockwise. (a) $\Gamma \sim 4$ cm^2/s, (b) $\Gamma \sim 11$ cm^2/s, (c) $\Gamma \sim 21$ cm^2/s, and (d) $\Gamma \sim 33$ cm^2/s. In both cases there is a dislocated wave front with a Burgers vector proportional to the vortex circulation. Water waves penetrate inside the vortex and the scattered wave is correspondingly asymmetric.

wave case. Thus, the predictions based on Eq. (1) coincide with experimental results in this respect.

In addition to the surface visualization just described, we have performed measurements of the amplitude and phase of the surface wave by measuring the deflection of a laser beam reflected at the wavy water surface [11]. Laser deflection is detected by a two axis position sensor. An x-y displacement system located just above the fluid provided the horizontal motion to scan the whole surface. Phase and amplitude of the wave can thus be measured with respect to the source by a lock-in amplifier, and this allows us to quantify the amount of dislocation by measuring the relative phase of the dislocated wave with respect to the incident one: The jump in phase obtained when crossing the line of dislocation is directly related to the adimensional Burger vector. By varying the wave frequency and vortex circulation we have verified that, in the low circulation Γ limit, α is proportional to both Γ and the inverse of group velocity c_g, as predicted by the theory.

The accuracy of the measurements obtained via laser beam deflection is quite high, and it allows us to test the limits of the theory used to derive Eq. (1). Figure 3 is an experimental spatial map of the wave amplitude for $\Gamma \sim 6$ cm^2/s, obtained by plotting at each point the maximum amplitude of the wave. Thus, this map does not contain phase information, and the variations in such amplitude in Fig. 3 correspond to nodal and antinodal lines resulting from the interference between the incident and the scattered wave. Note that dislocations are a result of a phase shift on the plane wave, so they cannot be detected with this method. The depression observed on Fig. 3 immediately behind the vortex is then not an interference effect but it is the shadow due to the finite size of the vortex core.

The information contained in Fig. 3 can be used to measure the scattering cross section of the surface wave by the vertical vortex. Indeed, this is obtained by measuring the surface deflection following the angular direction for a given distance from the vortex core. Figure 4 is the ratio of the scattered wave amplitude A_s to the incident amplitude A_0 as a function of polar angle θ, for several values of α and for a given frequency f. Here A_s is defined as $A_s = A - A_0$, where A is the total wave amplitude. Since oscillations in the measured amplitude A following the angular direction are due to constructive and destructive interferences between the incident and the scattered wave, the envelope of oscillations in A_s/A_0 is the scattering cross section which, as shown in Fig. 4, decreases with α. Theoretical values based on Eq. (1) are shown for comparison in Fig. 4a. Agreement is quite good for $\alpha = 0.5$ and becomes poorer as α increases. Now, the Burgers vector α is a linear increasing function of circulation Γ. In turn, a larger circulation determines a larger surface deformation

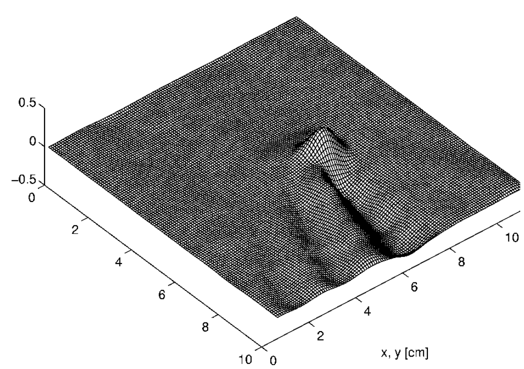

FIG. 3. Experimental surface deflection, obtained by scanning as described in the text, for $\Gamma \sim 6$ cm^2/s, $f = 18$ Hz, $\lambda = 1.1$ cm, and $c_g = 15.5$ cm/s. The elevation at the center is the surface deformation associated with the undisturbed vortex.

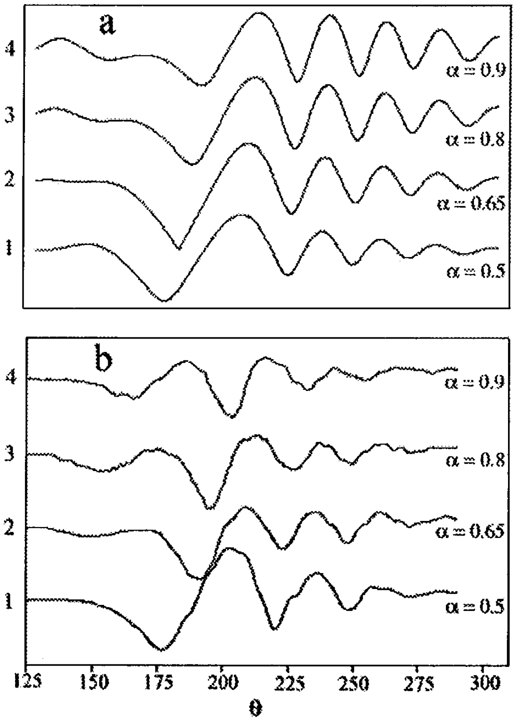

FIG. 4. Experimental surface wave amplitude A_s, normalized to A_0, obtained by scanning the water surface around the vortex following a circle of radius $r = 3$ cm, for $f = 18$ Hz, $\lambda = 1.1$ cm, $c_g = 15.5$ cm/s, and several values of the dislocation parameter α. Parts (a) and (b) show the theoretical and experimental results, respectively, for comparison.

associated with the undisturbed vortex core. Equation (1), being a leading order correction to the shallow water wave equation, considers wavelengths long compared to fluid depth, and does not consider the possibility of the penetration length of the wave into the fluid being of the same order as the undisturbed surface deformation associated with the vortex. We believe this fact may be at the root of the increasing disagreement between theory and experiment shown in Fig. 4 for increasing α.

In conclusion, we have measured the surface elevation of water waves in interaction with a vertical vortex, and analyzed it using an equation valid to leading order beyond shallow water theory. Although this takes into account dispersion only to a first approximation, the main experimental results are well captured by the theory: The incident wave front becomes dislocated, and there is an asymmetric scattered wave. Within experimental accuracy, and within assumptions needed to get a tractable theory, there is good agreement for the amount of dislocation as a function of vortex circulation, phase shift as a function of frequency, and scattering cross section as a function of circulation and wavelength. Detailed comparison with the scattering of a quantum mechanical electron by a solenoid into which it cannot penetrate shows that in both cases there appears a dislocated wave, but that in the latter case the impenetrable boundary conditions enforce a symmetry of the scattered wave. This symmetry no longer holds if the electron is allowed to penetrate inside the solenoid. Water waves can, of course, penetrate inside the vortex, and the scattering is correspondingly asymmetric.

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