

Embedded Solitons in Second-Harmonic-Generating Systems

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We present a new type of soliton, found in models characterized by opposing dispersions and competing nonlinearities at fundamental and second harmonics. They are isolated solitary waves, existing at discrete values of the propagation constant *inside* the system's continuous spectrum. We show analytically, and verify by simulations, that the fundamental solitons are linearly stable. They can be nonlinearly stable or unstable, depending on the sign of the energy perturbation, which could make these pulses useful for switching applications. Higher-order solitons are found, too, but they are linearly unstable.

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We report a new type of soliton, in the form of an isolated (*codimension-one*) solitary-wave solution whose intrinsic frequency resides *inside* the continuous spectrum of the radiation modes. It is a special member of a family of delocalized solitons, which are solitary waves with nonvanishing oscillating tails. In terms of the dynamical-systems theory, these are trajectories homoclinic to cycles, whereas ordinary solitons are homoclinic to fixed points. Delocalized solitons are known in various models of the hydrodynamic [1] and optical [2] origin.

We demonstrate that the amplitudes of the oscillating tails can exactly vanish at a discrete set of frequencies, resulting in a delocalized soliton becoming truly localized, and with finite energy. We call these solutions *embedded* (in the continuous spectra) *solitons* (ES). Because the vanishing of the tail's amplitude is an additional condition, these solitons (in contrast to familiar gap solitons [3]), never exist in continuous families, but only as isolated solutions.

A physical model giving rise to ES describes an optical medium with quadratic ($\chi^{(2)}$) and cubic ($\chi^{(3)}$) nonlinearities. Various systems of this type have been recently considered [4]. We start with a general one,

$$iu_z + (1/2)u_{tt} + u^*v + \gamma_1(|u|^2 + 2|v|^2)u = 0, \quad (1)$$

$$iv_z - (1/2)\delta v_{tt} + qv + (1/2)u^2 + 2\gamma_2(|v|^2 + 2|u|^2)v = 0. \quad (2)$$

Here, z and t are the propagation distance and reduced time [5], u and v are the fundamental- and second-harmonic (FH and SH) fields, $-\delta$ is the relative dispersion of SH, q is mismatch, and $\gamma_{1,2}$ are the Kerr coefficients. We will consider the case $\delta > 0$, $\gamma_{1,2} < 0$, which occurs in two different physical situations:

(i) anomalous and normal dispersion, respectively, at FH and SH (which is the most common case in optics), combined with self-*defocusing* cubic nonlinearity (that occurs in organic optical materials [6]) or (ii) the usual self-focusing Kerr nonlinearity, while the dispersion is normal at FH and anomalous at SH, which can be realized, e.g., in available media with two zero-dispersion points [5]. The recent progress in observing temporal solitons in $\chi^{(2)}$ media [7] suggests that the ES predicted in this work also might be observable. Typical numerical results for the model (1),(2) are given below for $\gamma_1 = \gamma_2 = -0.05$ and $\delta = q = 1$ (for other values, quite similar results have been obtained).

Stationary solutions to Eqs. (1) and (2) are sought for $u(z, t) = U(t) \exp(ikz)$, $v(z, t) = V(t) \exp(2ikz)$, where k is a propagation constant, U and V being real functions satisfying

$$(1/2)U'' - kU + U^*V + \gamma_1(U^2 + 2V^2)U = 0, \quad (3)$$

$$-(1/2)\delta V'' + (q - 2k)V + (1/2)U^2 + \gamma_2(V^2 + 2U^2)V = 0. \quad (4)$$

The linearization of Eqs. (1) and (2), with $u \sim \exp(ikz - i\omega\tau)$ and $v \sim \exp(2ikz - 2i\omega\tau)$, yields continuous spectra consisting of two disconnected branches,

$$k = -(1/2)\omega^2, \quad 2k = q + (1/2)\delta\omega^2. \quad (5)$$

If $q > 0$, there is a *gap*, $0 < k < q/2$, between them. Normally, a solution with k belonging to the continuous spectra cannot be localized, because there exist one or more modes with real ω , corresponding to a solution with

the following form at $t \rightarrow \pm\infty$: $U \approx b \exp(-\sqrt{2k}|t|)$, and

$$V \approx a \sin[\sqrt{(2/\delta)(2k - q)}t + \phi] + \dots, \quad (6)$$

where b , a , and ϕ are constants.

For a localized solution to exist, the independent exponential decay of the FH and SH tails in Eqs. (3) and (4) is not the only possibility. If instead, V and U^2 decayed at the same rate as $|t| \rightarrow \infty$, then the situation could be different, since now, the $\chi^{(3)}$ nonlinearity in (4) could be balanced against the V dispersion terms. This is an extension of the classical description of a soliton as a balance between dispersion and nonlinearity. Here, we have different nonlinearities balancing against different (FH and SH) dispersions.

Although solutions of (3) and (4) have been obtained numerically, there are analytically tractable systems demonstrating ES's. An example is the limit of Eqs. (1) and (2) when SH is much weaker than FH:

$$iu_z + (1/2)u_{tt} + u^*v + \gamma_1|u|^2u = 0, \quad (7)$$

$$iv_z - (1/2)\delta v_{tt} + qv + (1/2)u^2 + 4\gamma_2|u|^2v = 0, \quad (8)$$

which has the same continuous spectra as before. Exact single-humped soliton solutions to Eqs. (7) and (8) are

$$u = Ae^{ikz} \operatorname{sech}(\sqrt{2k}t), \quad v = Be^{2ikz} \operatorname{sech}^2(\sqrt{2k}t), \quad (9)$$

$$k = (1/2)(1 + 2\delta)^{-1}[q - (3\delta/2)(4\gamma_2 + 3\delta\gamma_1)^{-1}], \quad (10)$$

where $B = 2k(1 + 3\delta\gamma_1/4\gamma_2)$ and $A^2 = -3\delta k/2\gamma_2$. It is easy to show that for $\delta > 0$ and $\gamma_{1,2} < 0$, this is an ES solution (i.e., k belongs to the continuous spectra) if $(3\delta/2)(4\gamma_2 + 3\delta\gamma_1)^{-1} < q < -(3/4)(4\gamma_2 + 3\delta\gamma_1)^{-1}$.

Solutions of (3) and (4) were found numerically by the shooting method and Newton's iteration method. At certain discrete values of k , such that $k > q/2 > 0$ (i.e., *inside* the continuous spectra), the amplitude, a , of the tail (6) vanished, allowing the delocalized soliton to become a true ES. Two such values are $k_1 = 0.6963$ (see the corresponding ES in Fig. 1a) and $k_2 = 0.7136$, which gives rise to a double-humped ES (not shown here). If U and V components are both single humped, as in Fig. 1a, we call ES fundamental. Multihumped ES's were also found at other discrete values of k . Although the values of k which give rise to the exactly localized ES are found with a finite accuracy, their existence can be proved rigorously, because the continuous curve $a(k)$ crosses the zero axis. In the vicinity of any one of these values, say $k = k_1$, we may expand the amplitude of the tail as (a' is a constant)

$$a(k) = a'(k - k_1) + O((k - k_1)^2). \quad (11)$$

Note that ES's exist where ordinary solitons cannot occur. Hence the ES concept can considerably expand

the region of applicability of modern "soliton physics," including its promising applications to photonics. A crucial issue is the stability of an ES. A fundamental ES has no linear instability, while *all* ordinary solitons of (3) and (4), for the same parameters, are linearly unstable. These latter facts simply and clearly show the potential importance of an ES.

Before presenting the numerical results, it will be instructive to do a simple general analysis of ES stability. Since the ES is a member of a family of delocalized solitons, a generic small perturbation tends to move any ES over into an adjacent state, which always is a delocalized soliton with a nonzero tail. However, in this process, the solution's energy, $E = \int_{-\infty}^{+\infty} (|u|^2 + 2|v|^2) dt$, is conserved. The energy of an ES is finite, while any delocalized soliton has an infinite energy. Therefore, if the energy of the unperturbed ES, E_1 , plus the perturbation energy give a net initial energy $E_0 < E_1$, the energy lost in an attempt to build the infinite tail drives the soliton still farther away from the initial state. As a result, the perturbed ES can be expected to eventually decay into radiation. However, this is a slow *subexponential* decay (see below). On the other hand, if the perturbed state has energy $E_0 > E_1$, we can expect the energy lost through tail generation to drag the pulse back toward the unperturbed ES. Thus, the ES is subject to a subexponential *one-sided* instability, and we call such an object *semistable*.

To describe the semistability analytically, we note that the energy loss rate for a perturbed ES, as it attempts to build the tails of a small amplitude a , is

$$dE/dz = -Ca^2V, \quad (12)$$

where E is the energy of the pulse's central body, V is the inverse SH group velocity at given k , and C is a constant. We assume that, as the tail forms, the remaining central core of the soliton is the same as that of the delocalized one, allowing us to take k to be a function of its central core's energy E [in any particular model, $k(E)$ can be easily found numerically]. Thus, we may expand $k - k_1 \approx k' \cdot (E - E_1)$ [cf. Eq. (11)], where k' is a constant, and $k_1 \equiv k(E_1)$. Substituting this into Eq. (11), we obtain $d(E - E_1)/dz = -CV(a'k')^2(E - E_1)^2 + \dots$. The solution of this equation shows that the decay is indeed subexponential,

$$E(z) = E_1 + (E_0 - E_1)[1 + CV(a'k')^2(E_0 - E_1)z]^{-1}, \quad (13)$$

where E_0 is the initial energy of the perturbed pulse. When $E_0 > E_1$, $E(z) \rightarrow E_1$ as $z \rightarrow \infty$, i.e., the perturbed pulse approaches the fundamental ES. If $E_0 < E_1$, $E(z)$ at first slowly decays, then more rapidly, and finally disintegrating in a finite time.

To verify the predictions, we numerically simulated Eqs. (1) and (2). The numerical scheme used the pseudospectral method in the t direction and Runge-Kutta in

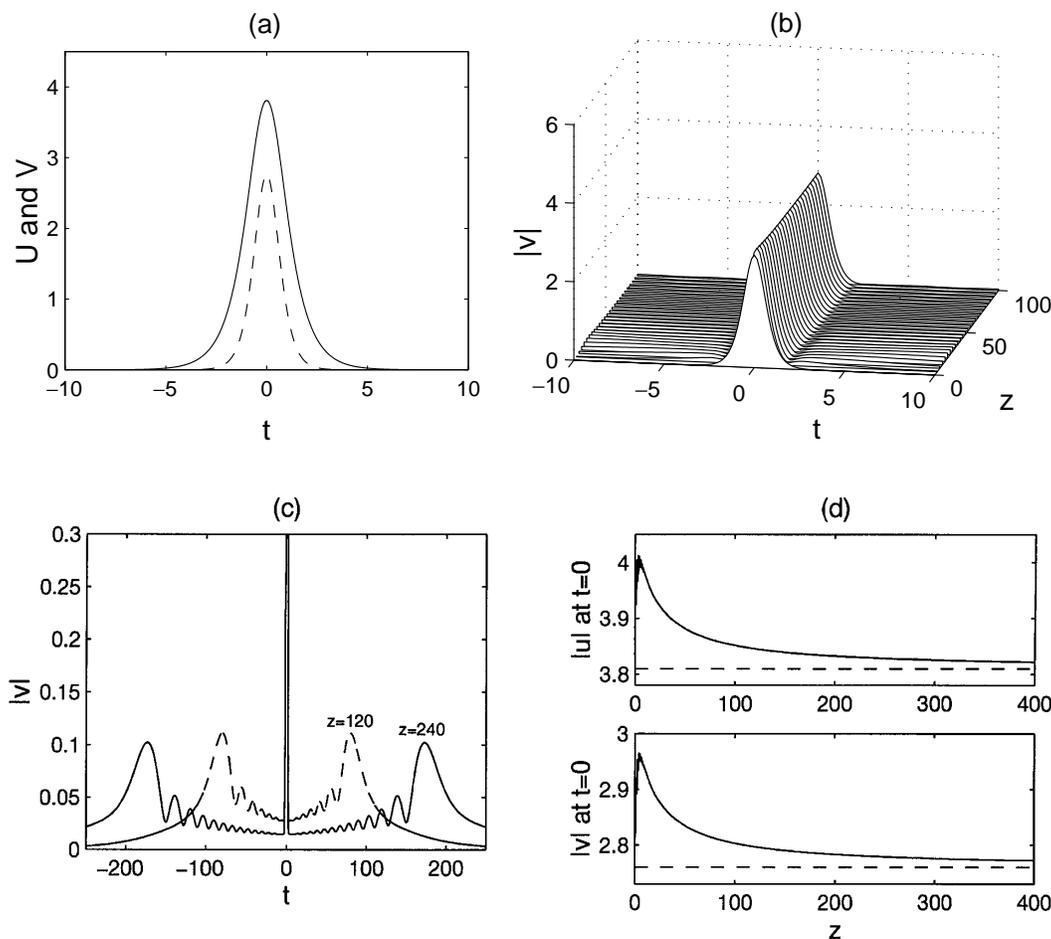


FIG. 1. (a) A typical example of a fundamental embedded soliton generated by Eqs. (3) and (4) at $\delta = 1, q = 1, \gamma_1 = \gamma_2 = -0.05$, and $k = 0.6963$; the solid and dashed curves show U and V . (b), (c), and (d): Evolution of the same soliton [only the SH component is shown except in (d)] whose initial energy was increased by the perturbation (14) with $c_1 = c_2 = 0.1$.

the z direction. The initial condition was

$$\begin{aligned}
 u(0, t) &= U(t) + c_1 \operatorname{sech}(2t), \\
 v(0, t) &= V(t) + c_2 \operatorname{sech}(2t),
 \end{aligned}
 \tag{14}$$

where (U, V) is the ES solution, and $c_{1,2}$ are small disturbance amplitudes. We first take the perturbation with $c_1 = c_2 = 0.1$, which yields $E_0 > E_1$. The result is shown in Fig. 1b. We see that, as z increases, the pulse slowly readjusts itself back to the original ES. Closer consideration shows that the perturbed v component tries to build tiny tails with a dominant frequency (relative to the variable t) $\omega \approx 0.88$. The dispersion relation (5) for SH yields $k \approx 0.6936$ for $\omega = 0.88$, which is very close to the above-mentioned wave number $k_1 = 0.6963$ of the unperturbed embedded soliton.

Next, we take the initial condition (14) with $c_1 = c_2 = -0.1$, which gives $E_0 < E_1$. The numerical evolution in this case is shown in Fig. 2. The perturbed ES initially decays slowly due to energy loss into the very small tails of the v component. The decay nonlinearly accelerates as the tail amplitudes grow. Eventually, the perturbed soliton perishes as predicted. After the decay, the u component completely disperses, while the v one evolves

into solitary pulses and radiation. This can be easily understood since Eq. (2) with $u = 0$ is the nonlinear Schrödinger equation that has its own solitons.

We stress that this semistability of the fundamental ES is different from and is weaker than a linear instability. As it follows from Eq. (13), an ES persists over the propagation distance $z_{st} \sim 1/[CV(a'k')^2(E_0 - E_1)]$, that can be transformed into $z_{st} \sim z_D[E_1/(E_1 - E_0)]$, where z_D is the soliton's dispersion length ($z \sim$ a few z_D , which may be as small as a few cm [7]), is sufficient to shape an arbitrary pulse into the soliton), and the second multiplier is the inverse of an initial relative energy disturbance. Obviously, one has $z_{st} \gg z_D$ for a sufficiently small initial disturbance; hence it should be possible to experimentally observe an ES (for instance, in a closed-loop system). Furthermore, our simulations have demonstrated that *all* ordinary (gap) solitons, for the same parameters in (3) and (4), are *exponentially* unstable. We have also numerically investigated the linearization of Eqs. (1) and (2) around the fundamental ES, finding no linearly unstable eigenmodes. Thus, the instability is indeed only nonlinear. Simulations of the higher-order (multihumped) ES verify that they quickly

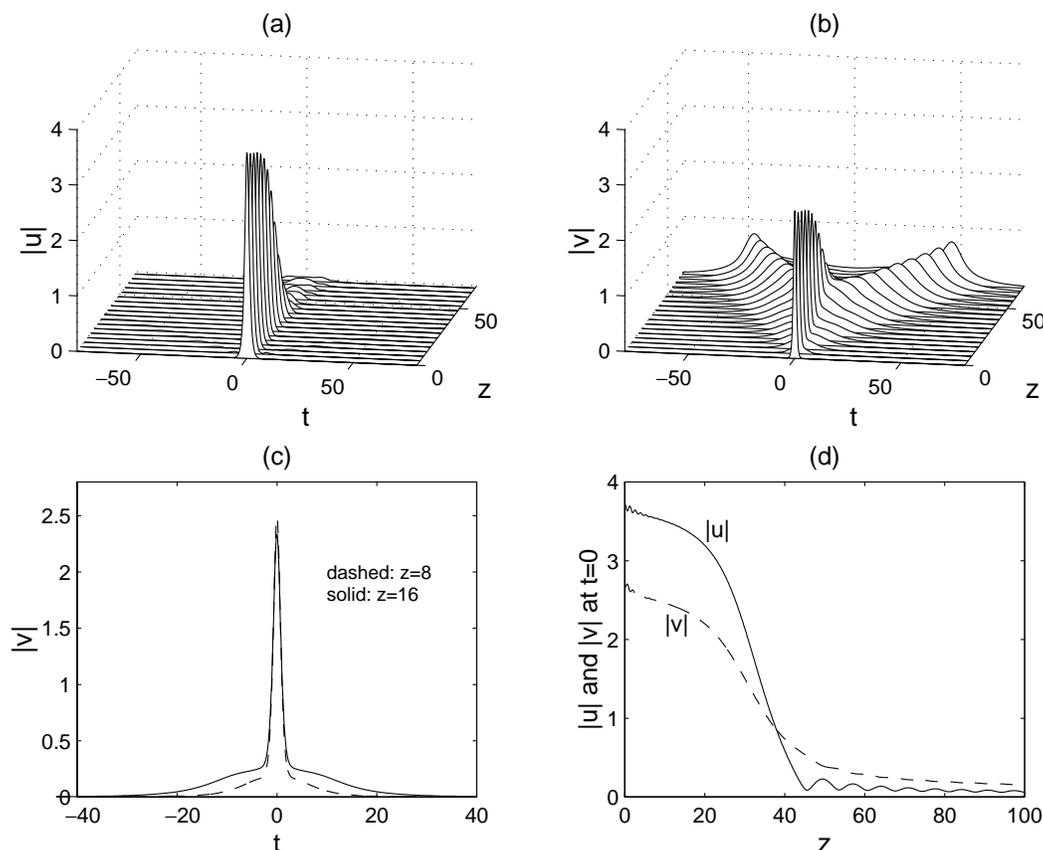


FIG. 2. Evolution of the perturbed fundamental embedded soliton from Fig. 1, whose initial energy was reduced by the perturbations (14) with $c_1 = c_2 = -0.1$.

break up, while the linearization about them reveals linearly unstable eigenmodes.

ES pulses may find applications as optical switches, since, if a pulse initially has an energy below E_1 , it decays, and in the opposite case, it stabilizes into an ES. Note also that, since the net energy crucially depends on the relative phase between the ES and the small perturbation, one can sense the phase of an ES pulse by means of a much weaker perturbing pulse, while the phase of ordinary optical solitons is unobservable.

The existence of the embedded solitons in Eqs. (1) and (2) is by no means a singular phenomenon. In fact, pulses found in the recent work [8] for the dispersive massive Thirring model are also ES. Very recently, a rich variety of ES solutions has been found in a three-component model with a purely quadratic nonlinearity [9].

The ES solutions considered here can be generalized to have a nonzero velocity, similar to what was done for ordinary solitons [10]. In particular, a very recent analysis of the above-mentioned dispersive Thirring model has demonstrated that one of the static-ES branches indeed bifurcates into a moving-ES one [9].

In conclusion, we have shown that a new class of isolated solitons will exist in models which contain mixed nonlinearities and dispersions. These solitons are linearly stable, and nonlinearly semistable. The sensitivity of

this semistable soliton to a test wave's phase may have potential for switching in optical devices.

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